INTRODUCTION

The Iris radio is a National Aeronautics and Space Administration (NASA) Deep Space Network (DSN)-compatible software-defined radio designed by the Jet Propulsion Laboratory (JPL) and built by the Space Dynamics Laboratory (SDL). Because of Iris’s small size and mass (0.5 L and 1.1 kg), it is ideal for small satellite or even CubeSat applications (Kobayashi et al., 2019). It has proven its utility for deep space missions, notably in the 2018 NASA MarCO mission, which flew a pair of CubeSats by Mars and provided real-time communications during NASA’s InSight landing using the Iris transponders (Klesh et al., 2018). Since then, the Iris radio has had numerous improvements, including the addition of regenerative ranging (Angkasa et al., 2019), which is primarily used to improve the performance of two-way pseudonoise (PN) ranging. These improvements have also made possible the addition of one-way range measurable on the uplink with only modest modifications to the Iris firmware. Recently, NASA’s CAPSTONE mission, launched on June 28, 2022 and currently near the Moon in a near-rectilinear halo orbit (NRHO), will perform navigation tests, including the use of one-way range and range rate for navigation (Thompson et al., 2022). In support of this, the Iris...
firmware has been updated to make measurements of one-way total count phase (to form the range rate) and one-way range that can be used for orbit determination and navigation.

Traditionally, deep space radios coherently transpond phase and range data to form two-way radiometric observables that are collected by ground-based transmitting and receiving stations (such NASA’s DSN or the European Space Agency [ESA]’s Estrack) and processed by ground-based space navigators. Classic examples of these transponders include General Dynamics’ Small Deep Space Transponder (Chen et al., 2000), which has been the workhorse transponder for many NASA missions over the past two decades, and Thales-Alenia’s Deep Space Transponder, which has a long heritage and is currently in use on ESA’s Bepi-Colombo mission to Mercury (De Tiberis et al., 2011). However, these radios are not able to formulate radiometric observables for use in onboard and autonomous space navigation. With its recent update, the Iris radio represents the first deep space radio capable of measuring and time-tagging both one-way Doppler and range data on the forward/uplink. In the current CAPSTONE experiment, the data will be telemetered to the ground for processing; however, this test will pave the way for future one-way radiometric tracking to be used as part of an onboard, autonomous navigation capability (Ely et al., 2021, 2022; Guinn et al., 2016).

The range and range rate data accuracy derived from Iris’s internal temperature-controlled crystal oscillator (TCXO) is not sufficient for many orbit determination applications; therefore, a more stable reference is needed to improve data accuracy. Recent research has shown that the use of a chip-scale atomic clock (CSAC) has the potential to produce satisfactory orbit determination performance for small satellite applications with modest navigation requirements in orbit at the Earth and Moon (Rybak et al., 2020, 2021). While a CSAC has better stability than a TCXO on times scales of tens of seconds or longer, its phase noise characteristics are worse than those of a TCXO and prevent its use as a reference oscillator for operating Iris. Rather, to improve the one-way range and range rate statistics using the superior long-term stability of CSAC, a one-pulse-per-second (1PPS) signal derived from a Microchip SA-45s CSAC is used by Iris to compare with the internal TCXO and to drive the collection of one-way range and range rate data.

In this work, the form of the one-way range and range rate data collected by Iris is derived, and methods are developed to translate the TCXO-derived range and range rate to their CSAC-derived counterparts. Two different methods, a direct computation method and a method that estimates clock differences using a Kalman filter, are developed for computing the one-way range rate data. The algorithms are applied to a multi-hour data set collected with an Iris/CSAC ground test setup, and the results are characterized. The analysis results show that the second method using a Kalman filter produces more accurate range rate results. The test results also show that the one-way range formed using CSAC-based timing remains more stable than its counterpart formed using TCXO-based timing. Therefore, it is recommended that the CSAC-based range rate computed using the second method and CSAC-based range be used for determining orbits over their TCXO-based counterparts, specifically for the navigation experiments to be performed by the CAPSTONE mission.

While the CSAC has superior long-term stability compared with the TCXO, it is still orders of magnitude less stable than the ground-based masers used to operate the DSN. As a result, the one-way radiometric data derived using the CSAC in this paper are also less precise by orders of magnitude than the ground-based two-way counterpart or one-way data derived from a more stable space clock, such as the Deep Space Atomic Clock. For this reason, the space navigation using these data will not be as accurate but is likely suitable for space missions with more modest requirements versus the more demanding needs of NASA’s larger missions.
2 | IRIS RADIO RECEIVER SIGNAL PROCESSING TO OBTAIN THE TOTAL COUNT PHASE

Iris is a superheterodyne receiver with a single down-conversion stage to an intermediate frequency (IF) of 112.5 MHz. A radio signal transmitted from ground and received by the Iris front-end can be modeled in the following real-valued form:

\[ r(t) = \sqrt{2P_T} \cos(\phi_{g,0} + 2\pi f_u (t - \tau(t)) - N_\phi + \beta \Phi_R (t - \tau(t))) + \nu(t) \]  

(1)

where we define the following:

1) \( t \) is the true (but unknown) spacecraft time. For the purpose of the current discussion, we set \( t_0 = 0 \) when Iris has locked and begins tracking the uplink carrier signal so that \( t \) represents an elapsed time (we will re-introduce \( t_0 \) later when we discuss the range measurement details).

2) \( P_T \) is the total received signal power.

3) \( \phi_{g,0} \) is the initial phase (in radians) of the signal transmitted by the ground antenna.

4) \( N_\phi \) is the phase ambiguity at the start of a tracking pass that represents the (unknown) number of wavelengths between the transmitting antenna and the receiving antenna.

5) \( f_u \) is the frequency of the uplink carrier and, for the NASA CAPSTONE mission, is set to 7,204,869,318 Hz.

6) \( \tau(t) \) is the time-varying one-way light time delay (i.e., the fundamental information relating to the astrodynamics of the navigation problem and the quantity we are seeking to isolate in the range and range rate measurements).

7) \( \Phi_R (t - \tau(t)) \) represents the ranging signal received by the Iris receiver at time \( t \) and transmitted by the ground station at time \( t - \tau(t) \). This term is a PN sequence that is a polar non-return to zero (NRZ) signal and is encoded using either the JPL (also known as DSN) or T4B range code (Massey et al., 2007; Consultative Committee for Space Data Systems, 2022). In the current case being examined, the chip rate of the sequence is \( f_c = (221/749)/1024 f_u = 2,076,044.255984542 \) Hz, and the associated chip period is \( T_r \approx 482 \text{ ns} \). The JPL code has been selected for the experiment on CAPSTONE.

8) \( \beta \) is the modulation index (and is set to zero when there is no ranging).

9) \( \nu(t) \) is a narrowband zero-mean, additive white Gaussian noise (AWGN) process that can be expressed in its in-phase and quadrature form:

\[ \nu(t) = \sqrt{2}v_i(t)\cos(2\pi f_u t) - \sqrt{2}v_q(t)\sin(2\pi f_u t) \]  

(2)

where the component processes \( v_i(t) \) and \( v_q(t) \) are both zero-mean white Gaussian noise, each with a two-sided power spectral density of \( S_{v_i}(f) = S_{v_q}(f) = N_0/2 \) and associated autocorrelation of \( R_{v_i}(h) = E[v_i(t)v_i(t+h)] = (N_0/2)\delta(h) \) and \( R_{v_q}(h) = R_{v_q}(h) \). The low noise amplifier (LNA) in the Iris front-end has a bandwidth of \( B_{IF} = 14 \) MHz centered around \( f_u \) that limits \( \nu(t) \) to this frequency range. The resultant band-limited noise \( \nu(t) \) has a variance of \( \sigma^2 = E[\nu^2(t)] = N_0 B_{IF} \) (Meyr & Ascheid, 1990).

The Iris local oscillator is a TCXO (Q-Tech Q804) with a specified nominal frequency \( f_0 \) of approximately 50 MHz. In practice, a physical oscillator will deviate
from its nominal frequency and can be described via the following phase and frequency relationship:

\[ \phi(t) = \phi_{t,0} + 2\pi f_0 \left( t + x_I(t) \right) \]  

(3)

where \( \phi_{t,0} \) is the initial phase. \( x_I(t) \) is the Iris TCXO’s deviation from true time \( t \) and is often called the clock phase offset. A physical oscillator’s clock phase offset can be modeled as follows:

\[ x_I(t) = x_{I,0} + \frac{\Delta f_{I,0}}{f_0} (t - t_0) + \frac{a_I}{2} (t - t_0)^2 + \frac{1}{2\pi f_0} (\psi_I(t) - \psi_I(t_0)) \]  

(4)

where \( x_{I,0} \) is the initial clock phase offset, \( \Delta f_{I,0} \) is the initial TCXO frequency offset from \( f_0 \), and \( a_I \) is the oscillator’s aging (or frequency drift). These parameters represent the oscillator’s deterministic deviations from a perfect oscillator relative to some defined reference time \( t_0 \). The term \( \psi_I(t) \) includes the stochastic phase effects of the oscillator (in radians). Again, without loss of generality, we will set \( t_0 = 0 \) and \( \psi_I(t_0) = 0 \). The model in Equation (4) is sufficiently general that it can apply to the CSAC as well; hence, a sub/superscript \( I \) will refer to Iris’s internal TCXO parameters, and a sub/superscript \( C \) will refer to the CSAC parameters. The real-time frequency of the TCXO can be obtained via the following:

\[ f_I(t) \triangleq \frac{1}{2\pi} \frac{d\phi}{dt} = f_0 \left( 1 + \frac{dx_I}{dt} \right) = f_0 \left( 1 + y_I(t) \right) \]  

(5)

where \( y_I(t) \triangleq dx_I/dt \) is the fractional frequency deviation:

\[ y_I(t) = \frac{\Delta f_{I,0}}{f_0} + a_I t + \frac{1}{2\pi f_0} \frac{d\psi_I(t)}{dt} \]  

(6)

Importantly, the model in Equation (4) is applicable when the radio environment has reached a steady-state operating temperature; for the testing and results examined in this paper, the radio has reached steady-state operating temperatures, and thus, Equation (4) is the relevant model. However, TCXOs and CSACs are also sensitive to temperature at levels that can affect the accuracy of radiometric measurements, which will likely be a factor to be examined during the CAPSTONE flight tests. Approaches for modeling dynamic temperature effects on one-way radiometric data were examined by Rybak et al. (2020), with results that could be of benefit to the future flight experiment. For the current ground testing, these models are not needed, but it is anticipated that they will need to be included when these future flight data are considered.

As shown in Figure 1, the Iris radio preprocesses the received signal \( r(t) \) by analog mixing down to an intermediate frequency \( f_{IF} \), digitizing the signal by sampling with the analog-to-digital converter (ADC), and then performs a digital down-conversion (DDC) to baseband. The baseband signal can now be tracked by the carrier tracking loop (CTL) to obtain the total count phase \( \hat{\phi}_I(t_m) \), which is used for forming the range rate. The CTL quadrature phase signal is routed to the symbol tracking loop (STL) to recover the fractional PN chip symbol timing \( \hat{\tau}_I(t_m) \) and to generate soft symbols \( \hat{\Psi}(k) \) used by the PN-sequence correlators to obtain the chip phase \( \varphi \) and Earth transmission time (ETT) of the received PN sequence. Figure 1 also shows the CSAC interface to Iris, which uses a 1PPS signal to sample the Iris radiometric telemetry and to obtain the relative timing between the Iris clock \( C_I(t) \) and the CSAC \( C_C(t) \). In the following sections, details of this
processing are described and models for computing the one-way uplink range rate and range are developed.

### 2.1 Analog Down-Conversion to the Intermediate Frequency

After receiving $r(t)$, the first step is to analog down-convert from $f_u$ to the intermediate frequency (IF) $f_{IF}$ (approximately 112.5 MHz) by mixing the received signal $r(t)$ with an Iris TCXO signal multiplied up to a specified frequency $\hat{f}_u$ and then bandpass-filtered (BPF) with a bandwidth $B_{IF} = 14$ MHz (the same as the LNA) centered around $f_{IF}$. The mixing frequency $\hat{f}_u$ is defined as follows:

$$\hat{f}_u = M_{IF} f_0$$  \hspace{1cm} (7)

where the value for $M_{IF}$ is selected such that the following holds:

$$f_{IF} = \left(f_u - \hat{f}_u\right) = \left(f_u - M_{IF} f_0\right) = 112.5 \text{ MHz} + \Delta f$$  \hspace{1cm} (8)

and $\Delta f$ is a known frequency bias. For example, NASA’s CAPSTONE mission has selected $M_{IF} = 2 \times 2766/39$ for the uplink carrier frequency $f_u = 7,204,869,318$ Hz to yield $\Delta f = 61,626$ Hz. Because the physical oscillator is not operating at exactly its nominal frequency $f_0$, an additional frequency term will be introduced in the down-conversion beyond $\Delta f$ that will be accounted for through the dependence on $x_I(t)$, as shown in Equation (3). The down-conversion to IF is performed via the following:

$$s(t) = \text{BPF}[r(t)\cos(\phi_{IF,0} + 2\pi \hat{f}_u (t + x_I(t)))]$$  \hspace{1cm} (9)

where $\phi_{IF,0}$ is the initial phase. By substituting Equation (1) into Equation (9) and reducing, the following is obtained:

$$s(t) = \sqrt{\frac{2P_T}{2}} \cos[\Delta \phi_0 + 2\pi (-f_u \tau(t) - \hat{f}_u x_I(t) + f_{IF} t - N_{\phi} + \beta \Phi_R(t - \tau(t))] + \nu_4(t)$$  \hspace{1cm} (10)
where the BPF process has eliminated higher-frequency terms that are outside $B_{IF}$. Here, we define $\Delta \phi_0 = \phi_{0,0} - \phi_{0,0}$. The resultant noise process $v_s(t)$ is a narrowband noise that has been shifted down from $f_u$ to $f_{IF}$ and takes the following form:

$$v_s(t) = -\frac{\sqrt{2}}{2} v_i(t) \cos \left( -\phi_{IF,0} + 2\pi \left( f_{IF} t - \hat{f}_u x_i(t) \right) \right)$$

$$-\frac{\sqrt{2}}{2} v_q(t) \sin \left( -\phi_{IF,0} + 2\pi \left( f_{IF} t - \hat{f}_u x_i(t) \right) \right)$$

(11)

The mixing process introduces a factor of $\frac{1}{2}$, and as a result, $v_s(t)$ has a reduced noise variance of $\sigma_v^2 = N_0 B_{IF} / 4$ (the component processes $v_i(t)$ and $v_q(t)$ still have the same two-sided power spectral density of $N_0/2$).

### 2.2 Analog-to-Digital Conversion and Alias Sampling

The next step is to convert the analog IF signal $s(t)$ to a digital signal and alias-sample it to a frequency of 12.5 MHz that prepares the signal for an efficient digital down-conversion to baseband. Digital sampling occurs on the TCXO cycle boundaries or, equivalently, a full cycle of its phase, which can be modeled by using Equation (3) as follows:

$$1 = \frac{\phi(t_n) - \phi(t_{n-1})}{2\pi} = f_0 \left( t_n - t_{n-1} \right) + f_0 \left( x_i \left( t_n \right) - x_i \left( t_{n-1} \right) \right)$$

(12)

Solving for $t_n$ yields the resulting difference equation:

$$t_n = t_{n-1} + T_s \left( x_i \left( t_n \right) - x_i \left( t_{n-1} \right) \right)$$

(13)

where $T_s = 1 / f_0$ is the period of the nominal sample step. Equation (13) reveals that, with a real oscillator, the actual time between samples varies non-uniformly (via the presence of $x_i(\cdot)$) and is an important artifact to track as we work to construct the range and range rate. If the oscillator is assumed to be perfect (i.e., $x_i(t) = 0$), then the discrete and continuous arguments reduce to their typical definitions. Solving the difference equation yields the following:

$$t_n = n T_s - x_i \left( t_n \right) \leftrightarrow n = \frac{1}{T_s} \left( t_n + x_i \left( t_n \right) \right)$$

(14)

where $n$ is the integral number of samples since the start of counting (recall that $t_0$ is defined to be zero). Note that the two equations in Equation (14) can be used to equivalently express a sampled quantity as a function of $n$ or $t_n$. Therefore, when it is convenient to identify the index of a function, we will use $n$, and when we want to identify the associated true time, we will use $t_n$. Substituting Equation (14) for $t$ in Equation (10) and rearranging yields the following:

$$s(n) = \sqrt{2P_r} \cos \left[ \Delta \phi_0 + 2\pi \left( -f_u \left( t_n \right) + x_i \left( t_n \right) \right) + nf_{IF} T_s - N_\phi \right]$$

$$+ \beta \Phi_R \left( t_n - \tau \left( t_n \right) \right) + v_s(n)$$

(15)
Now, the specific value for \( f_{IF} \), as defined in Equation (8), is selected to facilitate the alias-sampling according to the following identity:

\[
f_{IF} T_s = \frac{f_u - M_{IF} f_0}{f_0} = \frac{9}{4} + \frac{\Delta f}{f_0} = 2 + \frac{1}{4} + \Delta f T_s \tag{16}
\]

That is, with \( T_s = 1/f_0 = 20 \text{ns} \), we have \( 112.5/50 = 9/4 = 2 + 0.25 \), which will alias the signal from the IF 112.5 MHz to 12.5 MHz. Substituting Equation (16) into Equation (15) and rearranging the IF term leads to the following:

\[
s(n) = \frac{\sqrt{2 P_r}}{2} \cos \left[ \Delta \phi_0 + 2\pi \left( -f_u (\tau (t_n) + x_f (t_n)) + n \left( 2 + \frac{1}{4} + \Delta f T_s - N_\phi \right) \right)
+ \beta \Phi_R \left( t_n - \tau (t_n) \right) \right] + v_s (n) \tag{17}
\]

The following observation now proves useful:

\[
\cos \left( A + 2\pi n \left( 2 + \frac{1}{4} \right) \right) = \cos \left( A + \frac{2\pi n}{4} \right) \tag{18}
\]

Applying this relationship to Equation (17) produces an alias-sampled signal centered at \( \sim 12.5 \text{ MHz} \) (that is, \( f_s (\frac{1}{4} + \Delta f T_s) \)) that takes the following form:

\[
s(n) = \frac{\sqrt{2 P_r}}{2} \cos \left[ \Delta \phi_0 + 2\pi \left( -f_u \tau(t_n) - (f_u - \Delta f) x_f (t_n) + n \Delta f T_s + \frac{n}{4} - N_\phi \right) \right)
+ \beta \Phi_R \left( t_n - \tau (t_n) \right) \right] + v_s (n) \tag{19}
\]

Since \( n T_s \) is not readily known, whereas \( t_\phi \) can be estimated via the CSAC 1PPS and its comparison to the TCXO; we replace \( n T_s \) with \( (t_n + x_f (t_n)) \) (from Equation (14)) to yield, after some rearranging, the following expression for \( s(n) \):

\[
s(n) = \frac{\sqrt{2 P_r}}{2} \cos \left[ \Delta \phi_0 + 2\pi \left( -f_u \tau(t_n) - (f_u - \Delta f) x_f (t_n) + \Delta ft_n + \frac{n}{4} \right) - N_\phi \right)
+ \beta \Phi_R \left( t_n - \tau (t_n) \right) \right] + v_s (n) \tag{20}
\]

Similar steps are used to derive an expression for \( v_s (n) \) with the following result:

\[
v_s (n) = \frac{\sqrt{2}}{2} v_i (n) \cos \left( -\phi_{IF,0} + 2\pi \left( -f_u - \Delta f \right) x_f (t_n) + \Delta ft_n + \frac{n}{4} \right)
- \frac{\sqrt{2}}{2} v_i (n) \sin \left( -\phi_{IF,0} + 2\pi \left( -f_u - \Delta f \right) x_f (t_n) + \Delta ft_n + \frac{n}{4} \right) \tag{21}
\]

The digital sampling of the noise has changed the autocorrelation of the component noises into a discrete form \( \{v_i (n), v_q (n)\} \) with \( R_v (l) = R_q (l) = (N_0 / 2 T_s) \delta (l) \), and their associated two-sided discrete power spectral densities become \( S_{v_i} (e^{i 2\pi f T_s}) = S_{v_q} (f) = S_{v_q} (f) = (N_0 / 2 T_s) \) (Rice, 2009).
2.3 Digital Down-conversion and Lowpass Filtering

As illustrated in Figure 2, $s(n)$ is routed to the DDC where it undergoes a digital down-conversion in frequency to baseband via branching $s(n)$ into the in-phase arm (or I-arm) and quadrature arm (or Q-arm) signals, followed by mixing of the in-phase signal with $\cos(2\pi n / 4)$ and quadrature signal with $-\sin(2\pi n / 4)$. This yields baseband signals that are now down-sampled by every other value (i.e., $n = 2j$) and lowpass-filtered (LPF) to obtain the resulting I-arm and Q-arm signals $d_i(j)$ and $d_q(j)$, respectively. These signals can be represented in complex form as follows:

$$d(j) = d_i(j) + jd_q(j)$$

(22)

with $j = \sqrt{-1}$. The DDC is a traditional multiply–filter–decimate architecture, but has been implemented as a polyphase filter, yielding the multiply–decimate–filter architecture illustrated in Figure 2. The bandwidth of the lowpass filters is 25 MHz centered on 0 Hz and thus does not affect the $w(n)$ noise process that was previously band-limited to $B_{IF} = 14\text{MHz}$ by the LNA. In sampling every other value, Equation (14) can be rewritten to define a time argument $t_i(j)$ that applies to the I-arm as follows:

$$t_i(j) = j(2T_s) - x_i(t_i(j)) ↔ j = \frac{1}{2T_s} \left( t_i(j) + x_i(t_i(j)) \right)$$

(23)

where now the effective sampling period is doubled to $T_d = 2T_s$ and the associated sampling frequency is $f_d = 25\text{MHz}$. Similarly, a time argument for the Q-arm is defined as follows:

$$t_q(j) = (2j+1)T_s - x_i(t_q(j)) = t_q(j) + T_s + x_i(t_q(j)) - x_i(t_q(j))$$

(24)

Observe that the absolute sampling time $t_q(j)$ has been shifted by $\sim T_s$ relative to $t_i(j)$.

When interpreting the time argument results, it is important to note that any quantity that appears as a function of $t_i(j)$ or $t_q(j)$ implies that it is evaluated according to its original sampling by $t_n$ but selected from a list according to its absolute position in the shifted domain.
to its $j$-th argument. For example, the light time delay $\tau(t_n)$ yields a sequence \{\tau(t_0), \tau(t_1), \tau(t_2), \tau(t_3), \tau(t_4), \tau(t_5), \cdots\} that for $j = 0, 1, 2, \cdots$ and $\tau(t_{j}(j))$ results in a subset of the original sequence with values \{\tau(t_0), \tau(t_1), \tau(t_2), \cdots\} and, similarly, for $\tau(t_{q}(j))$ yields a subset with values \{\tau(t_1), \tau(t_2), \tau(t_3), \cdots\}. This implies that $d_i(j)$ and $d_q(j)$ have sampled slightly different values for $\{\tau(..., x_{j}(..., m(\cdots))\}$ and the associated component noise terms. Furthermore, as with the previous sampling that yields a sequence of values indexed by $n$ or values that are identified by their true time dependence by $t_n$, the time argument for a quantity using $j$, $2j$, or $t_{j}(j)$ and $t_{q}(j)$ or $2j+1$ is selected to clarify the relevant functional dependence of that quantity.

Let us begin by examining the I-arm signal $d_i(j)$, which is obtained via the following operation:

$$d_i(j) = \text{LPF}\left\{s(2j)\cos\left(\frac{2\pi(2j)}{4}\right)\right\}$$

(25)

Using $\cos(A + \pi j)\cos\pi j = \cos A$ and $\sin(A + \pi j)\cos\pi j = \sin A$ yields the following:

$$d_i(j) = \frac{\sqrt{2P_T}}{2} \cos\left[\phi_r(t_{i}(j)) + \beta \Phi_R\left(t_i(j) - \tau(t_{i}(j))\right)\right] + v_{di}(j)$$

(26)

where we have defined the total phase $\phi_r$ since the start of the pass as follows:

$$\phi_r(t) \triangleq \Delta \Phi_0 + 2\pi \left(-f_u \tau(t) - \left(f_u - \Delta f\right)x_i(t) + \Delta f t - N_u\right)$$

(27)

The in-phase noise $v_{di}(j)$ takes the following form:

$$v_{di}(j) \triangleq \frac{\sqrt{2}}{2} v_{i}(2j)\cos(\phi_r(t_{i}(j))) - \frac{\sqrt{2}}{2} v_{q}(2j)\sin(\phi_r(t_{i}(j)))$$

(28)

where, for notational convenience, we have defined the phase modulating the noise as follows:

$$\phi_r(t) \triangleq -\phi_{I,0} + 2\pi \left(-f_u - \Delta f\right)x_i(t) + \Delta f t$$

(29)

Because of the dependence on $\Delta f$, $v_{di}(j)$ is still a narrowband noise process. Similarly, the Q-arm signal $d_q(j)$ is obtained as follows:

$$d_q(j) = \text{LPF}\left\{-s(2j+1)\sin\left(\frac{2\pi(2j+1)}{4}\right)\right\}$$

$$= \frac{\sqrt{2P_T}}{2} \sin\left[\phi_r(t_{q}(j)) + \beta \Phi_R\left(t_q(j) - \tau(t_{q}(j))\right)\right] + v_{dq}(j)$$

(30)

where $\cos(A + \pi j + \pi/2)\sin(\pi j + \pi/2) = -\sin A$ and $\sin(A + \pi j + \pi/2)\sin(\pi j + \pi/2) = \cos A$ have been employed, and the quadrature phase noise $nu_{i,q}(j)$ is expressed as:

$$v_{dq}(j) \triangleq \frac{\sqrt{2}}{2} v_{i}(2j+1)\sin(\phi_r(t_{q}(j))) + \frac{\sqrt{2}}{2} v_{q}(2j+1)\cos(\phi_r(t_{q}(j)))$$

(31)

Careful examination of Equations (26) and (30) reveals that the signal has been converted to baseband with the total phase $\phi_r$ consisting of constants $\{\Delta \Phi_0, N_u\}$,
slowly varying terms such as the light time delay $\tau(t)$ and the clock phase offset $x_j(t)$, or the frequency bias $\Delta f$, which is much lower than the 25-MHz sampling frequency. In fact, the highest-frequency term is the ranging/data signal $m(t - \tau(t))$; however, this signal is also below 12.5 MHz (the Nyquist limit of the sampling frequency). We will see shortly how this term is separable from the phase terms in the CTL.

2.4 Carrier Tracking and Measuring the Total Count Phase

At this point, the Iris radio performs closed-loop phase tracking of the complex signal $d(j)$ from the DDC to generate estimates of the total count phase $\hat{\phi}_T(t_q(j))$ that will be used to form the one-way range rate measurements. Phase tracking is accomplished in the CTL, where the output of the CTL’s numerically controlled oscillator (NCO) produces the phase estimate. In addition to its use in phase tracking, the CTL isolates the ranging signal in the imaginary part of the tracking error signal and routes the signal to the STL for range demodulation to form range measurements. A block diagram of the CTL is shown in Figure 21 in Appendix A. The appendix also provides a detailed analysis of the CTL’s processing and resulting phase error variance.

The output of the NCO’s accumulation of phase, as given in Equation (93) in Appendix A, ensures that the phase is a total count that does not wrap frequently (until the 64-bit word length of the sampling register is reached) and enables sampling at a much lower rate. The register for the NCO’s phase accumulation is triggered for access by the Iris firmware via a 1PPS signal and reported out in telemetry on its 1-s boundaries. A continuous total phase is critical for forming range rate (or Doppler) measurements because the phase is sampled on a selected count time $T$ (typically greater than 1 s) and then differenced to compute a total phase change over $T$. If the phase wrapped on $2\pi$ boundaries, it would not be possible to form range rate measurements with sufficient precision for navigation. The 1PPS-sampled total count phase is represented as follows:

$$\hat{\phi}_T(t_m) = \Delta \phi(t_m) + \Delta \phi_0 + 2\pi \left(-f_u \tau(t_m) - (f_u - \Delta f)x_1(t_m) + \Delta ft_m - N_\phi \right) + \nu_{\phi_T}(t_m) \, \text{(rad)}$$

(32)

where $m$ is defined as the index associated with consecutive 1PPS boundaries (that is, $t_m - t_{m-1} \approx 1s$), the term $\Delta \phi(t_m)$ is the tracking loop error introduced by the CTL, which we will assume is negligible in later analysis, and $\nu_{\phi_T}(t_m)$ is the phase noise with an associated variance $\sigma_{\phi_T}^2$. A detailed development of the expression for the variance $\sigma_{\phi_T}^2$ is developed in Appendix A, with a final expression given in Equation (110). For convenience, we repeat the expression here:

$$\sigma_{\phi_T}^2 = \frac{N_\phi B_{CTL}}{P_c} (1 + 2R_\phi) + f_0 \left( h_2 f_0 + \frac{\pi \sqrt{2} h_0}{4 f_n} + \frac{\pi h_{-1}}{4 f_n} + \frac{\pi \sqrt{2} h_{-2}}{4 f_n} \right) \, \text{(rad}^2)$$

(33)

where $B_{CTL}$ is the CTL loop bandwidth, $P_c$ is the carrier power as defined in Equation (83), $R_\phi$ is the ranging signal-to-noise ratio defined in Equation (108), the coefficients $\{h_2, h_0, h_{-1}, h_{-2}\}$ are the Iris TCXO’s component noise spectral density strengths, which take the values given in Equation (106), and $f_n = 2\sqrt{2/3\pi} \cdot B_{CTL}$.
The Iris radio collects telemetry at a 1 Hz sampling rate with the relevant telemetry fields identified in Table 1 that are needed for forming the total count phase and, to be discussed later, the fields listed in Table 3 for range. The total count phase \( \hat{\phi}_T(t_m) \) is sampled on CSAC 1PPS boundaries (discussed in the next section) and recorded in telemetry, as defined in Table 1 for the total count carrier phase field. Before discussing the Iris range demodulation and formation of a range measurement, we turn to Iris timekeeping with the CSAC 1PPS and the formation of one-way range rate measurements.

### 2.5 Timekeeping on Iris with a CSAC 1PPS

Recall that the true time \( t_m \) is unknown; however, because the CSAC 1PPS input is available for comparison with the Iris TCXO, we can use the CSAC’s improved stability to reduce effects from TCXO drift and improve one-way radiometric data accuracy. The clock readings of the Iris radio derived from the TCXO \( C_I(t) \) and from the CSAC 1PPS \( C_C(t) \) can be represented as follows:

\[
C_I(t) = t + x_I(t) + \varepsilon_I(t) \\
C_C(t) = t + x_C(t) + \varepsilon_C(t) \tag{34}
\]

where \( x_I(t) \) and \( x_C(t) \) are the respective clock phases, \( \varepsilon_I(t) \) denotes the error in reading the Iris time, and \( \varepsilon_C(t) \) is the error in reading the CSAC time. The Iris time query errors \( \varepsilon_I(t) \) are at the sub-nanosecond level. For example, the absolute sampling jitter, obtained by integrating the TCXO phase noise profile in Equation (106), yields approximately 1.6 ps. Other clock query errors are also present; however, these turn out to be negligible when examining the use of the CSAC 1PPS to read the TCXO time. Indeed, the dominant error in using \( C_C(t) \) arises from the fact that the CSAC is asynchronous with the TCXO. That is, the CSAC 1PPS edge that triggers reading the TCXO will randomly fall somewhere within a TCXO sample interval (with a duration \( T_s = 20 \text{ ns} \)) and is representative of a quantization error. Other clock reading errors are present with the CSAC (for instance, the CSAC specification states that the 1PPS rising edge has a duration of <10 ns, representing another source of error); however, the quantization effect is the largest. Advanced models exist for characterizing quantization error, but a simple approach suffices in which this error is modeled as a zero-mean Gaussian white noise process with

<table>
<thead>
<tr>
<th>Field Name</th>
<th>Size</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1PPS Count</td>
<td>4 bytes</td>
<td>Running integral count of 1PPS seconds, which can accumulate up to ( 2^{32} ) seconds (or approximately 136 years)</td>
</tr>
<tr>
<td>Iris Time (seconds)</td>
<td>4 bytes</td>
<td>Count of seconds since Iris was turned on based on TCXO ticks; increments each time the subseconds counter rolls over</td>
</tr>
<tr>
<td>Iris Time (subseconds)</td>
<td>4 bytes</td>
<td>Count of 50-MHz TCXO ticks; rolls over to 0 after 49,999,999</td>
</tr>
<tr>
<td>Total Carrier Phase (TCP)</td>
<td>8 bytes</td>
<td>Integral number representing the total count phase since carrier lock; determines a floating-point estimate of phase cycles using ( \hat{\phi}_T(t_m) = \frac{C_I(t_m)}{2\pi} )</td>
</tr>
</tbody>
</table>

TABLE 1
Relevant Iris Telemetry Fields for Computing Time and Total Count Phase
variance $\sigma^2 = T^2_s/12$ (Lipshitz et al., 1992). The following applies for the CSAC 1PPS query noise $e_C(t)$:

$$T_s = 20\text{ ns} \rightarrow \sigma_C = 5.77\text{ ns}$$

Using the CSAC 1PPS rising edges as a trigger, samples are taken of the CSAC time $C_C(t)$, Iris time $C_I(t)$, total count phase $\phi(t)$ (for forming range rate measurements), and code phase $\Psi(q)$ (to be defined shortly and used for forming range measurements). In its raw form, the CSAC time $C_C(t)$ is measured as integer seconds since the time at which the radio was turned on (i.e., the 1PPS count field in Table 1), and the Iris time $C_I(t)$ has two parts: integer seconds and subseconds corresponding to cycle counts that can be converted into a fraction of a second by dividing by $f_0$ (i.e., the Iris time [seconds] and Iris time [subseconds] fields from Table 1, respectively). These data can be used to determine the Iris clock phase offset $x_I(t_m)$, where $m$ is now the count number associated with the consecutive 1PPS boundaries (that is, $m - (m - 1) = 1$, and is $-1$ s in duration). Using Equation (34), we obtain the following result:

$$x_I(t_m) \equiv C_I(t_m) - C_C(t_m) + x_C(t_m) + e_C(t_m)$$

where the error $e_I(t)$ has been neglected because $e_C(t_m) \gg e_I(t)$.

3 | FORMING THE RANGE RATE MEASUREMENTS

Equation (36) can be used in two different ways to formulate the one-way range rate. One method directly uses the total count phase in Equation (32), and the other method pre-filters the clock comparison data and calibrates the total count phase with the filter’s clock solution. These two approaches are described below.

3.1 | Method 1: Determining the One-Way Range Rate

Using Direct Calibration

Fundamentally, the range rate is obtained by differencing two total count phase values that are separated by a specified integration time; currently, these phase values are time tagged by the CSAC 1PPS and, as shown in Equation (32), has a direct sensitivity to the TCXO clock phase error. In this first approach, the impact of the TCXO is reduced by directly calibrating for the TCXO clock phase by substituting Equation (34) and Equation (36) into Equation (32), reducing, and rearranging, which leads to the following:

$$\dot{\phi}_I(t_m) = \Delta \phi_0 + 2\pi \left( -f_u \tau(t_m) + f_u C_C(t_m) - \left( f_u - \Delta f \right) C_I(t_m) \right)$$

Recall that $\Delta \phi(t_m) \approx 0$; thus, this term has been ignored. Moreover, in Equation (37), both $C_C(t_m)$ and $C_I(t_m)$ are recorded telemetry values. These
terms can be brought to the left-hand side and used to define a calibrated total count phase estimate $\overline{\phi}(t_m)$:

$$
\overline{\phi}(t_m) = \Delta\phi_0 + 2\pi \left( -f_{u}\tau(t_m) - f_u \left( x_{c}(t_m) + \epsilon_{c}(t_m) \right) - N_{\phi} \right) + \nu_{\phi}(t_m)
$$

(38)

where the right-hand side now contains the light time delay $\tau(t_m)$ (i.e., the central quantity of interest), a clock phase that is dependent on CSAC stability (rather than the TCXO with worse stability), the CSAC 1PPS clock query noise $\epsilon_{c}(t_m)$ (dominated by quantization error), the phase ambiguity $N_{\phi}$, and other phase error effects $\nu_{\phi}(t_m)$. It is worth noting that, while $\epsilon_{c}(t_m)$ is large in strength as compared with $x_{c}(t)$ at 1 s, the effect of $\epsilon_{c}(t_m)$ diminishes over longer time scales relative to $x_{c}(t)$ (which grows unbounded and is much larger than $x_{c}(t)$). This fact is the central motivation for calibrating the phase with the CSAC time comparison data.

We are now in a position to form an average one-way range rate measurement by differencing two calibrated phases that are separated in time by using the following definition:

$$
\dot{R}(t_m, t_{m-p}) \triangleq -\frac{c}{2\pi f_u} \frac{\Delta\overline{\phi}(t_m, t_{m-p})}{T} = -\frac{c}{2\pi f_u} \frac{\overline{\phi}(t_m) - \overline{\phi}(t_{m-p})}{T}
$$

(39)

where $\Delta f(t_m, t_{m-p}) \triangleq f(t_m) - f(t_{m-p})$ for any function $f(t)$ and $T = p$ seconds is the number of “seconds” between telemetry values recorded at time $C_{c}(t_m)$ and the values recorded $p$ samples earlier at $C_{c}(t_{m-p})$. We emphasize that $T$ is simply a defined value of integral seconds and acts as a scale factor to ensure that $\dot{R}(t_m, t_{m-p})$ has the correct units; it is not a real elapsed time. For instance, we could have just as easily defined a difference range measurement using $\Delta R(t_m, t_{m-p}) \triangleq -\frac{c}{2\pi f_u} \overline{\phi}(t_m) - \overline{\phi}(t_{m-p})$ that contains the same information as $\dot{R}(t_m, t_{m-p})$ without dividing by a time duration. Conveniently, the values of $p$ and, hence, $T$ equal the recorded interval of the CSAC 1PPS even though true time does not equal $T$ seconds (c.f., Equation (34)). By substituting Equation (38) into Equation (39), reducing, and rearranging, we obtain the following:

$$
\dot{R}(t_m, t_{m-p}) = c \left[ \frac{\Delta \tau(t_m, t_{m-p})}{T} + \frac{\Delta x_{c}(t_m, t_{m-p})}{T} \right] - c \left[ \frac{\Delta \epsilon_{c}(t_m, t_{m-p})}{T} + \frac{\Delta \nu_{\phi}(t_m, t_{m-p})}{2\pi f_u T} \right]
$$

(40)

where the ambiguity and initial phase offsets have been eliminated and all terms inside the square brackets are non-dimensional. The first bracket represents the quantities that are amenable to further estimation in the orbit determination filter (i.e., spacecraft position and velocity and CSAC clock parameters), whereas the second bracket contain elements that typically comprise the filter’s measurement noise. We make the following observations about the terms in order of their appearance:

1) $\frac{c}{T} \Delta \tau(t_m, t_{m-p})$ is the average range rate over time $T$ (or average Doppler shift if multiplied by $f_u$ instead of $c$) and contains the astrodynamics of interest plus other delay effects that modify the signal path length and introduce additional errors, such as atmospheric delays or solar plasma effects.
2) $\frac{c}{T} \Delta x_C \left( t_m, t_{m-p} \right)$ represents the average fractional frequency (in range rate) of the CSAC clock phase offset over time $T$. Note that the Iris TCXO clock phase offset has been eliminated from Equation (40) and has been replaced with the more stable CSAC clock offset. However, because the CSAC still drifts, orbit estimation should also include the components of the clock phase $x_C \left( t_m \right)$ as part of the estimation algorithm design. Moreover, because only a time difference appears in Equation (40), the estimation process will be relatively insensitive to the initial clock phase offset. However, because the CSAC time $C_{C} \left( t_m \right)$ is used to tag the measurement, some observability is present in the time tag. Although the filter will be estimating the CSAC components, it is instructive to assess the effect of the CSAC on the range rate for the short time scales typically used with range rate count times, $T < 600 \text{s}$. On these time scales, the CSAC error is dominated by white frequency noise, and its associated range rate uncertainty can be approximated via the following:

$$\sigma_{R_c} \left( T \right) = c \sigma_{\phi_C} \left( T \right) \cong c \frac{\sigma_{WF,C}}{\sqrt{T}}, \quad T < 600 \text{s}$$  \hspace{1cm} (41)

(A more precise expression for the Allan deviation is provided later in Equation (51) when Method 2 is discussed for computing the range rate.) From the CSAC specifications for the Microchip SA-45S, it can be determined that $\sigma_{WF,C} = 3 \times 10^{-10} \sqrt{s}$, which yields $\sigma_{R_c} \left( T \right) = 90 / \sqrt{T} \text{ mm/s}$. Relative to the other errors, this error decreases more slowly because it is proportional to $1 / \sqrt{T}$ vs $1 / T$.

3) $\frac{c}{T} \Delta x_C \left( t_m, t_{m-p} \right)$ represents the range rate error in reading the CSAC 1PPS leading edge time and is dominated by the quantization error. The uncertainty associated with this error takes the following form:

$$\sigma_{R_{s}} \left( T \right) = \frac{c}{\sqrt{6}} \frac{T_s}{T}$$  \hspace{1cm} (42)

which, for the Iris sample period $T_s$, yields $\sigma_{R_{s}} \left( T \right) = 2449.49 / T \text{ (mm/s)}$. For this test, and likely during operations, this is the largest error source dictating the precision of the one-way range rate derived using this method. In the results, this theoretical uncertainty will be compared with the measured results.

4) $\frac{c}{2\pi f_u} \Delta V_{\phi} \left( t_m, t_{m-p} \right)$ represents the range rate error from the phase noise remaining in the CTL, after it is locked and tracking the Doppler-shifted received phase. This term is the combination of the thermal phase noise related to the received signal strength, the ranging signal, and the oscillator that remains after the CTL is locked and tracking and has the phase variance given in Equation (33). For one-way links, the phase uncertainty is related to the range rate uncertainty as follows:

$$\sigma_{R_{s}} \left( T \right) = \frac{c\sqrt{2}}{2\pi f_u T} \int_{-\infty}^{\infty} \sigma_{\phi_s} \left( f \right) \frac{df}{f}$$

$$= \frac{c}{2\pi f_u T} \sqrt{\frac{N_o B_{CTL}}{P_c} \left(1 + 2R_S \right) + f_0^2 \left( h_2 f_0 + \frac{\pi h_0}{4f_n} + \frac{h_1}{2f_n^2} + \frac{\pi h_2}{4f_n^3} \right)}$$  \hspace{1cm} (43)

As an example, consider a link with a 40-dB margin over the minimum required $P_c / N_0 = 20 \text{ dB-Hz}$, thus $P_c / N_0 = 60 \text{ dB-Hz}$. For $B_{CTL} = 200 \text{ Hz}$ and
\[ \beta = 45^\circ, \text{ the resultant range rate uncertainty is } \sigma_{R_T}(T) = 0.23 / T \text{ (mm/s)}. \]

For \( P_c / N_0 = 40 \text{dB-Hz} \) (i.e., a 20-dB decrease in link strength), the range rate uncertainty increases to \( \sigma_{R_T}(T) = 1.33 / T \text{ (mm/s)}. \) Relative to the other terms discussed above, \( \sigma_{R_T}(T) \) is not significant; however, for traditional two-way data, this term can be a dominant error term. Moreover, if we examined the Iris one-way range rate without the CSAC 1PPS, this term would be significant.

There are potentially other errors, not delineated above, that should be included in Equation (40) if they are present; examples include multipath errors, phase wind-up errors, and other receiver errors.

3.2 Method 2: Determining the One-Way Range Rate Using Filtered Clock Phase Calibration

As noted in the prior section, quantization effects dominate the range rate errors when Equation (40) is used. It is possible to reduce the direct effects of quantization noise on the range rate calibration by first estimating the phase offset of the Iris TCXO relative to the CSAC by using the CSAC 1PPS/TCXO comparison data. This step requires introducing a clock model that can be used to form an estimation state vector. Let us consider the measurement equation for the clock difference data:

\[
\Delta C(t_m) \triangleq C_i(t_m) - C_C(t_m) = x_i(t_m) - x_C(t_m) - e_c(t_m)
\]

This expression can be cast into a state-space form as follows:

\[
\Delta C(t_m) = HX(t_m) - e_c(t_m)
\]

where the following definitions apply:

\[ H \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \ X(t_m) \triangleq \begin{bmatrix} x_iC(t_m) \\ y_iC(t_m) \\ y_tC(t_m) \end{bmatrix} \]

with a measurement noise strength of \( \sigma_{e_c} = 5.77 \text{ns}. \) A state-space model for the clock state \( X(t_m) \) can be formed by converting the continuous clock model defined in Equation (4) to a discrete model using an approach outlined by Zucca & Tavella (2005). In the present case, the clock model is now an aggregate of the CSAC clock phase \( x_C(t) \) and the TCXO clock phase \( x_i(t) \), where \( x_{IC0} \triangleq x_{I0} - x_{C0} \), \( \Delta f_{IC0}/f_0 \triangleq \Delta f_{I0}/f_0 - \Delta f_{C0}/f_0 \), \( a_{IC} \triangleq a_i - a_C \), and \( \psi_{IC}(t) \triangleq \psi_I(t) - \psi_C(t) \). The resulting linear time-invariant, three-state discrete clock model takes the following form:

\[
X(t_m) = \Phi(T)X(t_{m-p}) + G(t_{m-p})
\]

where

\[
\Phi(T) = \begin{bmatrix} 1 & T & T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}, \ G(t_{m-p}) = N(0, Q)
\]
where $T \triangleq p$ and the initial condition $X(t_0)$ is as follows:

$$X(t_0) \triangleq \left[ \begin{array}{c} x_{IC,0} \\ \Delta f_{IC,0}/f_0 \\ a_{IC} \end{array} \right] \quad (49)$$

The noise process $G(t_{m-p})$ is the discrete state-space form of $\psi_{IC}(t)/2\pi f_0$ and is distributed normally (i.e., Gaussian) with the following correlated covariance strength $Q$:

$$Q = \begin{bmatrix} \sigma_{WF}^2 T + \sigma_{RWF}^2 \frac{T^3}{3} & \sigma_{RWF}^2 \frac{T^2}{2} & 0 \\ \sigma_{RWF}^2 \frac{T^2}{2} & \sigma_{RWF}^2 T & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (50)$$

where $\sigma_{WF}^2$ is the noise strength of the oscillator’s white frequency noise component and $\sigma_{RWF}^2$ is the noise strength of the random walk frequency component. These noise strengths can be related to an oscillator’s Allan deviation $\sigma_y(T)$ at integration time $T$ as follows:

$$\sigma_y^2(T) = \frac{\sigma_{WF}^2}{T} + \frac{\sigma_{RWF}^2 T^2}{3} + \frac{a^2 T^2}{2} \quad (51)$$

Upper bounds for the aging and noise components of the Iris TCXO and CSAC can be obtained from their appropriate Allan deviation specifications (for the CSAC) or measured values (for the TCXO); using Equation (51) yields the values listed in Table 2. Applying these values from Table 2 to the aggregate noise process of the Iris TCXO/CSAC clock difference yields the following values:

$$\sigma_{WF,IC} = \sqrt{\sigma_{WF,I}^2 + \sigma_{WF,C}^2} = 3 \times 10^{-10} \sqrt{s},$$
$$\sigma_{RWF,IC} = \sqrt{\sigma_{RWF,I}^2 + \sigma_{RWF,C}^2} = 9 \times 10^{-13} \frac{1}{\sqrt{s}} \quad (52)$$

We now have enough information to form a Kalman filter estimate of the clock state $\hat{X}(t_m) = \mathbb{E}[X(t_m) | C(t_m)]$, where $C(t_m) = \{ \Delta C(t_m), \Delta C(t_{m-1}), \ldots, \Delta C(t_0) \}$. The conventional Kalman filter time update equations take the following form:

$$\hat{X}(t_m) = \Phi(T) \hat{X}(t_{m-1})$$
$$\hat{P}(t_m) = \Phi(T) \hat{P}(t_{m-1}) \Phi(T) + Q \quad (53)$$

and the Joseph forms of the measurement update equations are as follows:

$$K(t_m) = \hat{P}(t_m) H^T \left[ H \hat{P}(t_m) H^T + \sigma_{IC}^2 \right]^{-1}$$
$$\hat{X}(t_m) = \hat{X}(t_m) + K(t_m) \left[ \Delta C(t_m) - H \hat{X}(t_m) \right]$$
$$\hat{P}(t_m) = [I - K(t_m) H] \hat{P}(t_m) [I - K(t_m) H]^T + \sigma_{IC}^2 K(t_m) K^T(t_m) \quad (54)$$

Using a procedure outlined by Maybeck (1979), it is possible to find the steady uncertainty $P^{ss}$ of Equation (54) by approximating the Kalman filter with its
equivalent continuous form and solving for the steady solution. Doing so yields the following steady-state covariance (the subscript IC notation has been dropped):

\[
\mathbf{P}_{SS} = \lim_{t_m \to \infty} \left( \mathbf{P}(t_m) \right)
\]

\[
= \begin{bmatrix}
\sigma_{WF}^2 \sigma_z^2 \sqrt{T} & \frac{1 + 2 \sigma_z^2 \sqrt{T}}{\sigma_{WF}^2} \sigma_{RWF}^2 \sigma_z \sqrt{T} & 0 \\
\sigma_{RWF}^2 \sigma_z^2 \sqrt{T} & \frac{1 + 2 \sigma_z^2 \sqrt{T}}{\sigma_{RWF}^2} \sigma_{RWF}^2 \sigma_z \sqrt{T} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

(55)

where \( T \) is 1 s for the sampling interval of the 1PPS. Using the values defined above, we can predict the steady-state uncertainty for the TCXO/CSAC clock phase estimate to be \( \sigma_{\hat{x}} = \sqrt{\mathbf{P}_{SS}} = 1.38 \text{ ns} \), which, when compared with the quantization noise of 5.77 ns, represents a significant improvement in the uncertainty of the clock phase offset over the raw measurement.

We can use the real-time output of the Kalman filter to now form a calibrated range rate estimate. We begin by relating the Iris TCXO clock phase \( x_i(t_m) \) to \( \hat{x}_{\text{IC}}(t_m) \) as follows:

\[
x_i(t_m) = x_{\text{IC}}(t_m) + x_C(t_m) = \hat{x}_{\text{IC}}(t_m) + e_{x_{\text{IC}}}(t_m) + x_C(t_m)
\]

(56)

where \( e_{x_{\text{IC}}}(t_m) \) is the error in the estimate \( \hat{x}_{\text{IC}}(t_m) \) for \( x_{\text{IC}}(t_m) \), which has uncertainty \( \sigma_{\hat{x}_{\text{IC}}}(t_m) = \sqrt{\mathbf{P}_{xx}(t_m)} \). By substituting Equation (56) and Equation (34) into Equation (32), reducing, and rearranging, we obtain the following:

\[
\dot{\phi}_r(t_m) = \Delta \phi_0 + 2\pi \left( -f_u x_i(t_m) - f_u x_C(t_m) - \left( f_u - \overline{f} \right) \hat{x}_{\text{IC}}(t_m) + \overline{f} C_C(t_m) \\
- \left( f_u - \overline{f} \right) e_{x_{\text{IC}}}(t_m) - \overline{f} e_C(t_m) - N_\phi \right) + v_{\phi}(t_m)
\]

(57)
Observe in Equation (57) that $\hat{x}_{IC}(t_m)$ has just been computed and $C_C(t_m)$ is a recorded telemetry value; thus, both terms can be brought to the left-hand side to define a calibrated total count phase estimate $\bar{\phi}(t_m)$ as follows:

$$\bar{\phi}(t_m) = \Delta\phi_0 + 2\pi \left[ \int_{u} f_u \tau(t_m) - \int_{u} f_u x_C(t_m) - \int_{u} f_u e_{x_w}(t_m) - \int_{u} f_u e_C(t_m) - N_\phi \right] + \nu_{\phi}(t_m)$$

(58)

As before, we determine the one-way range rate by substituting Equation (58) into Equation (39) to obtain the following:

$$\dot{R}(t_m, t_{m-p}) = c \left[ \frac{\Delta\tau(t_m, t_{m-p})}{T} + \frac{\Delta\nu(t_m, t_{m-p})}{T} \right]$$

$$- c \left[ \frac{\Delta f}{f_u} \frac{\Delta x_C(t_m, t_{m-p})}{T} + \frac{\Delta \nu_{\phi}(t_m, t_{m-p})}{2\pi f_u T} \right] + \frac{\Delta f}{f_u} \frac{\Delta e_C(t_m, t_{m-p})}{T}$$

(59)

where the terms in the first square bracket match the prior result in Equation (40) and the second square bracket has a slightly different set of errors. The errors with differences from Method 1 are now described.

1) $\frac{c}{T} \left( 1 + \frac{\Delta f}{f_u} \right) \Delta x_C(t_m, t_{m-p})$ represents the range rate error introduced by using the estimate $\hat{x}_{IC}(t_m)$ and has a magnitude that can be estimated from the steady-state uncertainties as follows:

$$\sigma_{\dot{R}}(T) = c \sqrt{2 \left( 1 + \frac{\Delta f}{f_u} \right) \sigma_{\dot{x}}^{ss}}$$

(60)

For $\sigma_{\dot{x}}^{ss} = 1.38 \text{ns}$, we obtain $\sigma_{\dot{R}}(T) = 585 / T \text{ (mm/s)}$. As will be seen with the experimental test results, this estimate is too large by a factor of 5 at $T = 1 \text{ s}$ but matches the test results on longer time scales.

2) $\frac{c}{T} \frac{\Delta f}{f_u} \Delta e_C(t_m, t_{m-p})$ represents the range rate error in reading the CSAC 1PPS leading edge time and is dominated by the quantization error; however, the magnitude of this term is greatly reduced relative to the term appearing in Equation (40) because the coefficient now contains only the fractional frequency bias, which is $\Delta f / f_u = 8.6 \times 10^{-6}$, in the present case. The uncertainty for this term is as follows:

$$\sigma_{\dot{R}}(T) = c \sqrt{\frac{\Delta f}{f_u} \frac{T_s}{T}}$$

(61)

and takes the value $\sigma_{\dot{R}}(T) = 0.02 / T \text{ (mm/s)}$. Relative to the prior term, this is not a significant effect and can be neglected.

4 | IRIS RADIO RECEIVER SIGNAL PROCESSING TO OBTAIN RANGE

We turn now to developing the range measurement model. As noted in Equation (82) in Appendix A, when the CTL is phase-locked, then $\Delta \phi(t_y(j)) = 0$. 


and, referring to Figure 21, the Q-arm error signal Im{e(j)} becomes dominated by the ranging term \( \frac{2j}{m} \left( t_q(j) - \tau(t_q(j)) \right) \) (whereas the I-arm signal, when locked, remains proportional to the carrier via \( \cos \Delta \phi(t_q(j)) \equiv 1 \)). Consequently, Im{e(j)} is routed to the ranging demodulator for extracting the range measurement (i.e., correlate on \( R(t - \tau(t)) \) to obtain a measure of \( R(t) = \alpha(t) \)). Notably, when the CTL is phase-locked and tracking the Im{e(j)} signal, it does not have the phase delay due to the Doppler shift (i.e., \( f_\tau \tau(t) \)); therefore, no additional compensation is needed for the ranging to accommodate Doppler shifts.

As described by Hamkins et al. (2016), the PN range signal \( \Phi_R(t) \) is a polar NRZ signal of a composite code comprised of six separate component codes (defined using either the JPL or T4B codes) that, when combined, have a period of \( L = 1,009,470 \) chips. The range signal \( \Phi_R(t) \) is defined as follows:

\[
\Phi_R(t) = \sum_{q=-\infty}^{\infty} \Psi(q)p(t - qT_r - \epsilon_r)
\]

where \( \Psi(q) \in [-1, 1] \) is the composite PN range code, \( p(t) \) is a unit-amplitude rectangular pulse of duration \( T_r \) (i.e., the chip duration), and \( \epsilon_r \) is an unknown symbol timing epoch that is assumed to be uniformly distributed in the interval \(-T_r/2 \leq \epsilon_r \leq T_r/2\). The associated chip rate is \( f_r \approx 2,076,044 \), hence, we have a duration of \( T_r \approx 482\text{ ns} \). The code period is \( T_{PN} \approx 0.486\text{ s} \), which yields a range ambiguity of approximately 145,800 km. The composite range code consists of a range clock \( \Phi_{RC}(t) \) that oscillates between 1 and –1 with a period of two chips and then a logical construction with the five remaining codes, which have periods of 7, 11, 15, 19, and 23 chips. If we examine only the range clock, it defines a rectangular waveform as a function of time that conforms to the following:

\[
\Phi_{RC}(t) = \text{sgn}(\sin(2\pi f_{RC}t))
\]

where \( f_{RC} = 1/(2T_r) \) is the frequency of the range clock waveform. An important feature of the composite code is its strong correlation with the range clock:

\[
R_{\Phi_R, \Phi_{RC}}(0) = E[m(t)m_{RC}(t)] = \begin{cases} 0.96, \text{ JPL code} \\ 0.93, \text{ T4B code} \end{cases} \quad \text{(Massey et al., 2007)}
\]

A result of this high correlation when using either the JPL or T4B code is that the ranging demodulator can use a scheme that first locks onto the range clock phase using the Iris STL to obtain the fractional code phase offset \( \epsilon_r \) and then correlates on the other component codes to determine the integer index of the PN range code phase, that is, \( q \) from \( \Psi(q) \).

### 4.1 Symbol Timing Recovery for Range

Iris utilizes its STL for both two-way regenerative ranging and one-way uplink ranging to recover an estimate of the symbol timing \( \hat{\epsilon}_r \) and to generate the soft symbols \( \hat{\Psi}(k) \) that are sent to the ranging correlator (the index \( k \) is the time index of the STL after the integrate-and-dump [I&D] step that operates at the chip rate \( f_R \)). A block diagram of the STL is shown in Figure 22, with details of its operation and estimates of the symbol timing variance \( \sigma^2_{\hat{\epsilon}_r} \) provided in Appendix B: Iris symbol timing variance. With the STL locked, Iris records the PN chip accumulator (PCA)
telemetry defined in Table 3 (i.e., this is the output of the phase accumulator in Figure 22), where the fractional chip phase $\hat{\epsilon}_r$ can be estimated as follows:

$$\frac{\hat{\epsilon}_r(t_m)}{T_r} = \frac{\text{PCA}}{2^{27}} - \text{int} \left( \frac{\text{PCA}}{2^{27}} \right)$$  \hspace{1cm} (65)

The estimate has the associated variance $\sigma_{\epsilon_r}^2$ given by Equation (129) in Appendix B. For convenience, we repeat the expression for the variance $\sigma_{\epsilon_r}^2$ here:

$$\sigma_{\epsilon_r}^2 = T_r^2 \left[ \frac{N_0 B_{STL}}{2 P_r} \left( 1 + \frac{3}{8 R_s} \right) \right]$$  \hspace{1cm} (66)

where $B_{STL}$ is the STL loop bandwidth, $P_r$ is the ranging power as defined in Equation (83), and $R_s$ is the ranging signal-to-noise ratio defined in Equation (108).

### 4.2 | Forming the Range Measurement

The light time delay $\tau(t)$ of the received range code $m(t – \tau(t))$ can be defined as follows:

$$\tau(t) \triangleq t - t_E$$  \hspace{1cm} (67)

where $t_E$ is the epoch associated with the ETT and is uniquely associated with the current chip $\Psi(q)$ that defines $m(t – \tau(t))$ as defined in Equation (62). Because the PN sequence has a period of approximately 0.5 s, there will be an ambiguity associated with the measured delay; as a result, the light time can be decomposed further as follows:

$$\tau(t) = \Delta \tau(t) + N_r T_{PN}$$  \hspace{1cm} (68)
where $N_r$ is an integer ambiguity associated with the “wavelength” of the range code and $\Delta \tau(t)$ is a fractional portion of the range code. Because the range code is a repeating sequence, the following property holds:

$$
\Phi_R(t - \Delta \tau(t)) = \Phi_R(t - \Delta \tau(t) + N_r T_{PN}) = \Phi_R(t - \Delta \tau(t))
$$

(69)

where the range signal $\Phi_R(t - \Delta \tau(t))$ is the actual signal that Iris has locked onto and is tracking.

After obtaining the fractional phase of the received chips $\hat{\phi}_r$, the STL routes the soft chips $\Psi(k)$ to the component code correlators for acquisition; details of this process have been described by Angkasa et al. (2019) and Hamkins et al. (2016). Once $\Psi(k)$ has been acquired and locked, the value of the index $q$ in Equation (62) is determined (i.e., the location $q$ of the chip $\Psi(k)$ as an element of the sequence $\Psi(k-q)\Psi(k+1-q)\ldots\Psi(k)\ldots\Psi(k+L-q)$) by correlating on the component codes to determine the individual delays from local reference component codes. These individual delays are then combined via the Chinese remainder theorem (Hamkins et al., 2016) to determine a value for $q$, which is reported in telemetry as the PN chip index from Table 3. As the CSAC 1PPS samples the recorded value of $q$ and the fractional phase of the chip $\hat{\phi}_r$, the following associations can be made from the telemetry:

$$
\Phi_R(t - \Delta \tau(t)) \leftrightarrow q + \frac{\hat{\phi}_r(t_m)}{T_r} \leftrightarrow C_i(t_m) \text{ and } C_c(t_m)
$$

(70)

The next important step in determining a value for the light time delay $\tau(t)$ is to relate the received code to the ETT as recorded in the telemetry (i.e., the ETT and ETT epoch fields in Table 3). In this first test of uplink one-way ranging with the DSN and CAPSTONE, the ETT will consist of two elements. The first element is a precursor command sent at the beginning of a ranging pass that identifies the UTC epoch on a second boundary. The train of PN sequences begins transmission at the UTC epoch on its second boundary to within 1/nobreakspace 100 m. For our analysis, we will assume that the ground transmitter frequency reference has perfect stability; therefore, the transmitted period of each chip $T_r$ can be treated as exact.

Now, each transmitted PN sequence has a reserved field of 32 bits that overwrite the first 32 chips of the sequence; however, bit 31 encodes the PN epoch bit that would have appeared in the zeroth position. These bits encode the integer counter position of the PN sequence that has been an element of the sequence train since transmission started. Hence, the first sequence would have 0, the second 1, and so on. With this, the ETT integer value can be combined with the epoch in the precursor command to determine the time that any selected PN sequence has been transmitted (note that this approach eliminates the range ambiguity $N_r$ as an unknown). Because the start of transmission has an error of ±1 µs (or 300 m), only one PN epoch should be used to initialize the ETT; otherwise, the time of transmission of each sequence would be effectively white noise with an uncertainty of 1 µs, eliminating its utility for orbit determination. Using only one epoch to initialize the ETT, the following can be applied to determine a value for $t_E$ using the recorded telemetry:

$$
\hat{t}_E(t_m) = t_{E_0} + T_r \left( \frac{ETT \times L + q + \frac{\hat{\phi}_r(t_m)}{T_r} + \delta}{t_{E_0} + \frac{T_r}{T_r}} \right) + b_E
$$

$$
= t_{E_0} + \frac{T_r}{T_r} \left( \frac{ETT \times L + q + \frac{\hat{\phi}_r(t_m)}{T_r} + \delta}{T_r} \right) + v_c(t_m) + b_E
$$

(71)
where \( \hat{t}_E(t_m) \) is the computed \( t_E \), \( t_{E_0} \) is the epoch identified in the precursor command, \( b_E \) is an unknown (bias) error in the realized \( t_E \), \( ETT \) is the integer obtained from the telemetry and determined by associating the ETT epoch to the PN epoch that identifies the specific PN sequence associated with \( q \), and \( \hat{\epsilon}_r(t_m) = \epsilon_r(t_m) + \nu_c \) is the fractional chip phase determined using the PCA (see Table 1 for descriptions of these fields). As noted in Appendix B, the STL is operating with the \( T_d = 2T_s \) sampling period, but it is possible to recover the full resolution of the \( T_s \) sampling by noting the following:

\[
\delta = \begin{cases} 
0, & \text{Iris subsecond is even} \\
i & T_s \text{, Iris subsecond is odd} 
\end{cases}
\]  

(72)

Finally, recall that we defined the time \( t \) as an elapsed time past some epoch \( t_0 \). We now re-introduce the initial epoch \( t_0 \) corresponding to the true epoch of when the radio clock began counting. Via the operational spacecraft clock comparison process, an estimate of \( \hat{t}_0 \) can be found and is expressed as follows:

\[
\hat{t}_0 = t_0 + x_0
\]  

(73)

where \( x_0 \) is the initial clock phase offset identified in Equation (4) and represents a random bias error with initialization of the spacecraft clock for epoch \( \hat{t}_0 \). We are now able to form the uplink one-way range measurements. First, we derive a range measurement using the Iris TCXO as the reference:

\[
R_I(t_m) = c(C_I(t_m) - \hat{t}_E(t_m)) = c(\tau(t_m) + x_I(t) - b_E) + c(\epsilon_r(t) - \nu_c(t_m))
\]  

(74)

Second, using the superior stability of CSAC as the reference, we find the following:

\[
R_C(t_m) = c(C_C(t_m) - \hat{t}_E(t_m)) = c(\tau(t_m) + x_C(t) - b_E) + c(\epsilon_r(t) - \nu_c(t_m))
\]  

(75)

Note that \( x_0 \) has been absorbed into the respective clock expressions, \( C_I(t_m) \) or \( C_C(t_m) \). The expressions for \( R_I(t_m) \) and \( R_C(t_m) \) are grouped similarly to their range rate counterparts, where the terms involving the light time delay \( \tau(t_m) \), \( x_C(t_m) \), and \( b_E \) will contribute to the estimation state defined for the orbit determination filter while the noise terms are uncompensated, additive errors. The associated range error variances for the noise terms are given as follows:

1. \( c\epsilon_r(t_m) \) is the error associated with sampling the TCXO. Using the associated jitter calculation, the error has an uncertainty of \( \sigma_{\epsilon_r} = c(1.6 \, \text{ps}) = 0.0048 \, \text{m} \)
2. \( c\epsilon_C(t_m) \) is the error associated with using the CSAC 1PPS for the sample TCXO time; this term is dominated by the quantization error as identified in Equation (35) and has an uncertainty of \( \sigma_{\epsilon_c} = 1.73 \, \text{m} \)
3. \( \nu_c(t_m) \) is the STL symbol timing error contributing to the range error in Equation (66) and takes the following form:

\[
\sigma_{R_{\text{STL}}} = c\sigma_{\epsilon_c} = cT_s \sqrt{\frac{N_0 B_{\text{STL}}}{2P_r} \left( 1 + \frac{3}{8R_s} \right)}
\]  

(76)

To elucidate the performance of the STL at recovering range, it is useful to plot Equation (76) as a function of \( P_r/N_0 \) and for various loop bandwidths \( B_{\text{STL}} \),
as shown in Figure 3. The test results shown later use an operating range with $B_{STL} = 10$ Hz and $P_r / N_0 = 60.8$ dB-Hz, which yields a range uncertainty of 0.38 m (1σ). However, during flight, the expected $B_{STL} = 200$ Hz and $P_r / N_0 \in (50.8, 60.8)$ dB-Hz yield an expected range uncertainty between 11.4 m for the lower power limit and 1.7 m for the higher limit (1σ).

The ETT bias error $cb_E$ will likely change from pass to pass, and the orbit filter will be required to estimate its value. Additionally, there are uncertain path delays between modulating the signal on the carrier and exiting the feed horn of the transmission antenna. This additional error can vary from a few meters to many tens of meters. A conservative a priori uncertainty for $cb_E$ is $c\sigma_E = 320$ m (representing a root sum square [RSS] of the path length error and the modulation time error) and should be reset on each pass. Finally, recall that the initial clock phases $x_{I,0}$ and $x_{C,0}$ have been resolved via ground to spacecraft commanding to an accuracy of 1 ms (equivalently, 300 km). This, like $b_E$, is a bias error that will be estimated by the orbit filter but, unlike $b_E$, will be a constant for the duration that the radio remains on. Methods for improving the ETT bias calibration and initializing the onboard spacecraft clock are a topic of future investigation.

5 | RANGE RATE RESULTS FROM THE TWO METHODS

We now compare the range rate results for the two methods using data collected from an Iris radio test conducted on May 5, 2021. The test had the following configuration:

1) Uplink constant carrier frequency derived from a maser with $f_u = 7,204,869,318$ Hz,
2) CSAC providing a 1PPS input to the Iris radio that triggers data sampling at a rate of 1 Hz,
3) Q-Tech Q804 TCXO as the Iris local oscillator, and
4) Data collection for over 24 h (89,257 s), with the analysis focused on the last 8 h of data to ensure that the radio and oscillators had reached thermal equilibrium.

The difference between the Iris TCXO time and the CSAC time (i.e., the raw time measurement of Equation (36)) is shown in Figure 4 on the left. On the right is
the second difference (i.e., the rate), which exhibits a trimodal characteristic that reflects the combination of effects from quantization error and the relative drift of the TCXO and CSAC. Note that after approximately 81,000 s, the size of the second difference shifts upward, indicative of a frequency drift between the TCXO and CSAC.

The raw total count phase in cycles (i.e., Equation (37)) collected during the test is shown in Figure 5. Because there is only a constant carrier frequency (no ramps or Doppler shift), the expected response is roughly linear and representative of the frequency bias \( \Delta f \) between the uplink carrier and the IF as well as the TCXO frequency deviation from the nominal \( f_{0,y_I}(t) \).

### 5.1 Method 1 – Direct Phase Calibration Using Equation (38)

We start by examining the first method, which uses direct calibration for TCXO effects; the Allan deviation (as determined using an overlapping Allan deviation...
[OAD] computation) of the raw total count phase $\phi(t_m)$ (normalized to time deviations by dividing by $f_u$) and the calibrated phase $\bar{\phi}(t_m)$ computed from Equation (38) (normalized with $(f_u - \Delta f)$) are shown in Figure 6. Also shown is the OAD of the raw total count phase with its deterministic drift removed. With a constant carrier and no Doppler shift, this normalization will result in the OAD of the raw phase, representing the stability of the TCXO, and the calibrated phase, representing the stability of the TCXO/CSAC combination (which includes quantization effects). Note that the raw phase OAD at 1 s is $4 \times 10^{-11}$, whereas the specification for the TCXO at 1 s is $< 1 \times 10^{-9}$, indicating that the actual TCXO in the Iris radio has a much better stability than specified. After approximately 100 s, the raw phase has a slope that is proportional to $\tau$ (the Allan deviation integration time – not to be confused with the light time delay or the range rate integration time), which is representative of the oscillator’s frequency drift. This growth is the motivation for attempting to calibrate using the CSAC 1PPS, which exhibits better stability on longer time scales. However, on time scales under 1,000 s, the OAD shown in Figure 6 for the calibrated phase is dominated by quantization effects (as indicated by a slope proportional to $1/\tau$). Indeed, the OAD at 1 s is $~ 6 \times 10^{-9}$ and, when multiplied by the speed of light, yields a rate of 1,800 mm/s, which is close to the range rate measured at 1 s via Method 1. After 200 s, the calibrated phase crosses over the raw phase OAD curve. This cross-over point represents the required integration time needed for the associated one-way range rate derived using the calibrated phase to begin having improved noise characteristics as compared with the range rate derived from the raw phase.

The range rate is computed via Equation (40) for the integration times $T \in \{1,10,30,60,90,120,150,180,300\}$ s and, as expected, results in a precision that is inversely proportional to the integration time, as indicated by Equation (42) (and, to a lesser degree, by Equations (41) and (43)). Recall that the predicted dominant uncertainty arises from the quantization error effect when the CSAC is used in comparison with the TCXO, as given by Equation (42). The short-term noise from the TCXO and the thermal noise are much smaller than the quantization effect. These uncertainties and the associated standard deviation of the measured range rate are shown in Figure 7 on a log–log scale, where the range rate statistics have...
been computed after the removal of any deterministic bias, rate, and acceleration terms, as determined via a least-squares fit (i.e., “detrending”).

It is clear from the apparent linear slope of the experimental results that the uncertainty is proportional to $1/T$ (after approximately 10 s) and is consistent with the theoretical results given by Equations (41)–(43). The most significant deviation between the experimental and theoretical results arises for the fundamental 1-s integration time, where the predicted theoretical range rate uncertainty is 2,451 mm/s and the experimental result is 1,797 mm/s, corresponding to a 27% difference. For all other measured count times, the percent difference is within ±6.5%, with a mean of only –0.005%. This agreement indicates that the theoretical one-way range rate uncertainties given in Equations (41)–(43) provide a reasonable prediction of the actual uncertainties.

Another measure of the range rate precision obtained using Method 1 is the OAD computed using the range rates (now normalized by dividing by the speed of light) for different integration times, as shown in Figure 8. By comparing the curves in Figure 8 with the OAD of the normalized calibrated total count phase
in Figure 6, we observe that the OADs for the normalized range rates are close to those of the normalized phase. This result is expected (because the normalized range rate is equivalent to the fractional frequency derived from the normalized phase) and shows that the range rate calculations are correct.

It is useful to examine the results for integration times of $T = 1\, \text{s}$, which is the fundamental data collection interval, and $T = 60\, \text{s}$, which is the typical integration time used in deep space navigation. Results for the 1-s integration time are shown in Figure 9, with the top plot exhibiting the range rate computed from the raw total count phase, the middle plot showing the range rate computed from Equation (40), and, as performed for the results in Figure 7, the bottom plot showing the calibrated range rate with the bias, rate, and acceleration effects removed or “detrended” as determined by a least-squares fit. The range rate in the top plot illustrates the effect of a drifting TCXO with nearly a 50-Hz reduction in the frequency bias over 8 h. Additionally, the combined effect of $\Delta f$ and the TCXO frequency deviation can be seen, with a value near 57 kHz. The range rate in the middle plot represents the data that would be used for orbit and clock determination, where the clock is now effectively the CSAC rather than the TCXO. The detrended curve removes the deterministic effects of the CSAC and can be used to derive the range rate noise.
characteristics and the associated standard deviation shown in Figure 7. Notably, we observe the trimodal distribution of the range rate and its similarity to the second difference in Figure 4.

The range rate values from the bottom, detrended plot are shown as a histogram in Figure 10. The bulk of the data lies at the center, and the estimated kernel density (dashed black line) suggests conformity to a uniform-density distribution at the center (a consequence of being dominated by quantization error). As the integration time increases, the sampling begins to approach a normal distribution, as will be seen for the next integration time of $T = 60$ s.

Turning to $T = 60$ s, the computed values for the three cases (raw, calibrated, and calibrated/detrended) of the range rate are shown in Figure 11, and an associated histogram of the calibrated/detrended data is shown in Figure 12. In the middle and bottom plots in Figure 11, the range rate data appear unimodal with a more normal distribution. This distribution is confirmed in Figure 12, where the estimated kernel density and fitted normal distribution are well matched, indicating that the distribution is indeed normal. Using this approach yields a one-way range rate uncertainty of 40.5 mm/s ($1\sigma$), which is 400 times larger than the typical two-way range rate of 0.1 mm/s provided by standard DSN services to a typical deep space transponder. This result motivates examination of the second method.

### 5.2 Method 2 – Filtered Phase Calibration Using Equation (58)

In this method, a clock phase estimate $\hat{x}_{IC}(t_m)$ is obtained via a Kalman filter and used to calibrate the total count phase as in Equation (58). We begin by examining the properties of $\hat{x}_{IC}(t_m)$; in this case, the prefit residuals are the clock difference data $\Delta C(t_m)$, and the postfit residuals are computed from $\Delta C(t_m) - H\hat{x}(t_m)$. These residuals are shown in Figure 13, with the prefit residuals at the top and the postfit residuals at the bottom.

Recall from the differenced clock data in Figure 4 that, at roughly 81,000 s, there is an upward shift of the clock differences, which results from the relative clock
drift between the TCXO and CSAC combined with the quantization effect of the clock measurements. This shift is the source of the transition seen in the postfit residuals during the same period (with the x-axis now shifted by 60,675 s and now occurring near 21,000 s) and reflects the Kalman filter reacting to the increasing magnitude of the clock differences. Later, we will see that this transition impacts the noise levels on the calibrated range rate.

The mean of the postfit residuals is 0.42 ns, and the associated standard deviation is 5.61 ns, resulting in a root mean square of 5.62 ns. We can compare this result to the theoretical postfit residual uncertainty derived from the Kalman filter, which is 

\[ \sigma_{\text{ss}} = \sqrt{\left( \sigma_{e_{\text{IC}}} \right)^2 + \left( \sigma_{x_{\text{IC}}} \right)^2}, \]

The actual steady-state solution uncertainty \( \sigma_{x_{\text{IC}}} \) from the filter is 1.34 ns (compare to the prediction in Equation (55) of 1.38 ns) and \( \sigma_{e_{\text{IC}}} = 5.77 \) ns; together, these values yield a theoretical steady-state postfit residual uncertainty of 5.92 ns, which compares favorably with the measured value of 5.62 ns. The OADs of the prefit residuals, postfit residuals, actual filter clock solution, and normalized raw total count phase are shown in Figure 14.

The clock phase estimate \( \hat{x}_{\text{IC}}(t_m) \) (green curve) is the Kalman filter estimate of the TCXO/CSAC difference. The OAD curve of the raw total count phase

**FIGURE 11** Range rate results for Method 1 with \( T = 60 \) s; range rate from the uncalibrated phase (top), calibrated range rate (middle), and calibrated range with bias, rate, and acceleration effects removed (i.e., “detrended”) (bottom)
normalized by \((f_u - \bar{f})\) (red curve) is a representation of the actual TCXO phase \(x(t_m)\) AD in this test configuration; therefore, when the green curve is closer to the red curve, this indicates a better filter estimate. Clearly, \(\hat{x}_{IC}(t_m)\) accurately captures the drift characteristics of \(x(t_m)\) after approximately 100 s (that is, the CSAC drift contribution is not significant for \(x_{IC}(t_m)\) or \(\hat{x}_{IC}(t_m)\)). Prior to that time, the solution approaches the actual TCXO phase \(x(t_m)\), but the effects of quantization errors (i.e., the measurement noise in the filter) and CSAC short-term noise prevent full recovery. The largest difference occurs at 1 s, with an OAD for \(\hat{x}_{IC}(t_m)\) of \(2.5 \times 10^{-10}\) while the OAD for \(x_{IC}(t_m)\) is \(4 \times 10^{-11}\); however, at 100 s, the difference becomes insignificant. Even with these differences, \(\hat{x}_{IC}(t_m)\) can be used as a reasonable proxy for \(x_{IC}(t_m)\) to reduce the range rate errors introduced by the TCXO drift and the effects of quantization noise, as compared with Method 1.
As a final comment, the OAD of the postfit residuals is approximately white phase noise because its slope is close to $1/T$, indicating that the Kalman filter is producing an optimal estimate for $x_{IC}(t_m)$.

Using $\hat{x}_{IC}(t_m)$ to calibrate the total count phase according to Equation (58), we compute the associated OAD of the raw total count phase $\hat{\phi}(t)$ and the calibrated phase $\hat{\phi}(t_m)$, with results shown in Figure 15. The calibrated phase results obtained using Method 2 are shown in green, and for comparison, the prior Method 1 results are shown in red. Also shown is the raw total count phase with its deterministic drift removed. Both raw results – regular and drift-removed – are the same as in Figure 6 because they represent the same data.
A careful examination of the OAD results for $\tilde{\phi}(t_m)$ in Figure 15 reveals much better performance (i.e., lower Allan deviation values) for Method 2 versus Method 1 on all integration times below 5,000 s, and after 5,000 s, the results are commensurate. Hence, Method 2 produces better phase stability than Method 1. The $\tilde{\phi}(t_m)$ OAD results from Method 2 for integration times of 1–55 s lie above those generated by the TCXO; for longer integration times, the calibrated phase has lower OAD values than the TCXO and exhibits characteristics representative of the CSAC.

Turning to the range rate calculation of Equation (59) for Method 2 and for integration times of $T \in \{11, 03, 06, 09, 120, 150, 180, 300\}$ s, we obtain the measured range rate uncertainties shown in Figure 16 as the blue curve, labeled “Method 2 exp.” Similar to our analysis of Method 1, we compare these data with the theoretical result; for Method 2, the theoretical result is Equation (60), shown as the orange dash–dot curve. We also compare these results with the experimental results from Method 1, shown as the green curve.

The theoretical results for Method 2 with a 1-s integration are larger than the experimental results by a factor of 5, but on longer time scales, the theoretical results match the experimental results more closely. After 40 s, the experimental results for Method 2 begin to slightly exceed the theoretical results; it is suspected that this is the effect of the CSAC instability (i.e., the second term in Equation (59)) beginning to affect the range rate uncertainty. In fact, this instability may be dominant relative to the $\sigma_{\text{RIC}}(T)$ term. For instance, at 100 s, the specified CSAC stability is $3 \times 10^{-11}$. This stability can be converted to a range rate by multiplying by the speed of light, which results in an uncertainty of 9 mm/s. The interpolated 100-s experimental range rate uncertainty for Method 2 in Figure 16 is 8–9 mm/s, which presents a very good comparison and suggests that CSAC errors are beginning to dominate. In this figure, the experimental range rate crosses over the theoretical result at $T = 30$ s, where the uncertainty is ~18 mm/s. Converting to an Allan deviation yields $6 \times 10^{-11}$, which is commensurate with the specified CSAC Allan deviation levels shown in Table 2. Therefore, after 30 s, the range rate errors begin to be dominated by CSAC effects. Compared with Method 1 (experimental results shown in green), Method 2 produces smaller range rate uncertainties for all
time scales investigated. From this, we can conclude that Method 2 is preferable for deriving more accurate one-way range rate measurements.

In Figure 17, we plot the OADs computed from the range rates based on Method 2 (normalized by the speed of light) for selected integration times. These OAD results can be compared with the calibrated phase OAD results based on Method 2, as shown in Figure 15; as expected, the results match closely, indicating that the one-way range rate calculations are correct.

We now show the specific range rate results obtained via Method 2 for integration times of $T = 1$ and 60 s. To facilitate this comparison, the raw, calibrated, and calibrated plus detrended range rates are shown for both integration times in Figure 18, as well as associated histograms. We start by examining the results for $T = 1$ s; relative to Method 1, which showed three discrete distributions, Method 2 produces a unimodal distribution with a standard deviation of 101 mm/s, 18 times smaller than the equivalent standard deviation from Method 1. Superimposed on the distribution is the associated normal distribution, and a deviation is observed relative to the estimated kernel density. Indeed, the kernel density has a shape that more closely matches a uniform distribution, suggesting that the quantization effects are still dominant for this integration time. In the results for $T = 60$ s, the estimated kernel density more closely matches the associated normal distribution, indicating that the statistics of the range rate errors are becoming more characteristic of Gaussian white noise. This is an important result in that the associated orbit determination filters typically treat the measurement noise as Gaussian white noise; therefore, the range rate results for $T = 60$ s match the orbit determination assumptions better than those for the shorter integration time. This was also true for the range rates obtained via Method 1 at $T = 60$ s, but the standard deviation for Method 2 is only 11 mm/s (compared with a standard deviation of 41 mm/s for Method 1). As noted previously, the effect of the CSAC (at least as specified\(^3\)) begins to dominate at $T = 30$ s. This trend implies that one-way range

\(^3\)The conclusion that the CSAC begins to dominate (vs. other effects such as the quantization error, which falls off proportional to $1/\sqrt{T}$) at $T = 30$ s is derived from the CSAC specification values; the actual CSAC likely performs better than specified. If this is the case, then the integration time at which the CSAC begins to dominate would shift to the right. However, the qualitative effect that the CSAC would begin to dominate the range rate noise is still valid.
rates computed using Method 2 begin to have accuracies that are similar to those that would be derived if the Iris radio used the CSAC as its local oscillator rather than the TCXO.

Finally, in the calibrated range rate results for $T = 60$ s shown in Figure 18, the residuals become noisier at a time point near $22,000$ s. This noise is an artifact of the quantization effects seen in the delta time residuals of Figure 13, where the differences switch from the bottom to the top.
6 | RANGE RESULTS

The range measurement analysis is based on a laboratory test conducted on October 5, 2021 that ran for ~25 h. The test configuration was similar to the range rate test conducted previously. The significant difference was that the signal power was increased to ensure that the STL remained locked on the range sequence. Indeed, with the range modulation index set to $\beta = 45^\circ$, the resultant range signal-power-to-noise ratio was $P_r/N_0 \approx 60.8$ dB-Hz. Additionally, the STL noise bandwidth was set to $B_{STL} = 10$ Hz. As with the range rate tests, a sufficient duration at the start of the test was used to ensure that all elements thermally stabilized (particularly the oscillators); as a result, the useful data and statistics begin at 40,996 s into the test.

Recall from Section 4.2 that the ETT requires information uplinked from the ground to form an effective one-way range measurement. Because the focus of this test was analyzing time intervals for forming range so that we could assess the statistics and stability of the ranging process, no association with a wall clock time for the ETT was needed. The ETT was simply a monotonically increasing time count that was computed in accordance with Equation (71) but with $t_E^0 = 0$ and $b_E = 0$ as well. Similarly, both the Iris time and CSAC time were processed based on their recorded time durations from telemetry; no association with a wall clock time was needed. The range data were computed using either Equation (74) for range measurements referenced to the Iris TCXO (i.e., $R_I(t_m)$) or Equation (75) for range measurements referenced to the CSAC (i.e., $R_C(t_m)$). Issues relating to ground calibration of $b_E$ or initialization of $x_{I,0}$ or $x_{C,0}$ with actual epochs were not addressed with this testing, and when either $b_E$, $x_{I,0}$ or $x_{C,0}$ were present in the data they were zeroed out.

Figure 19 shows the raw (left, top) and detrended (left, bottom) Iris TCXO-based range $R_I(t_m)$ and the raw (right, top) and detrended (right, bottom) CSAC-based

![Figure 19 Iris TCXO range results: raw (upper left) and detrended (lower left); CSAC range results: raw (upper right) and detrended (lower right)](image-url)
As with the range rate results, detrending consists of performing a least-squares fit of the data to estimate an initial offset, rate bias, and acceleration bias. Because the range signal that is being tracked by the STL removes all of the internal frequency biases and Doppler shifts, the remaining range dynamics in this test configuration can be directly related to the deterministic and stochastic effects of the oscillator and the associated symbol timing and quantization errors. The least-square estimates for the deterministic fractional frequency and aging terms of the Iris TCXO and CSAC are delineated in Table 4 in both a time format and a resultant distance relationship. Note that, compared with the range rate results, the TCXO drift for this test is slightly lower (8.1 × 10^{-9}/day vs. 2.4 × 10^{-8}/day) and illustrates the variability that is possible even for a single oscillator. Additionally, the observed CSAC aging is larger than the specified value of 3 × 10^{-11}/day.

Despite the variation observed for these parameters, they can be determined as part of the orbit determination process and would likely be elements of the orbit filter’s state vector. With the effects of these parameters removed from the range data (bottom plots in Figure 19), the stochastic response of the recovered range can be better ascertained. Notably, the Iris TCXO-based range exhibits a random walk with a variation of ~1,500 m at the end of the test, while the CSAC-based range has much less systematic variation (several meters) but an increased short-term noise relative to the TCXO. The difference in these short-term noise characteristics, as well as longer-term noise, is best ascertained by examination of the OAD, as shown in Figure 20.

![Figure 20](image)

**FIGURE 20** OAD of the raw CSAC range (blue) and raw Iris TCXO range (orange), which have both been normalized to time by dividing by the speed of light; CSAC (green) and TCXO (red) OAD derived from total count phase measurements.

<table>
<thead>
<tr>
<th>TABLE 4</th>
<th>Estimated Fractional Frequency Offsets and Aging for the Iris TCXO and CSAC Determined From Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iris TCXO</td>
<td>( \frac{M_f}{T_0}, \frac{\ddot{R}}{T_0^2} = 6.07 \times 10^{-7} \text{ s/s}, 1821 \text{ m/s} )</td>
</tr>
<tr>
<td>CSAC</td>
<td>( 1.50 \times 10^{-9} \text{ s/s}, 0.45 \text{ m/s} )</td>
</tr>
</tbody>
</table>
The raw CSAC-based range is shown in blue, whereas the raw TCXO-based range is shown in orange. For comparison, the OADs for the CSAC (green) and TCXO (red), as derived from the total count phase data, are also shown. Recall that the calibrated phase from Method 2 in Equation (58) yields the best measure of the CSAC response, with the primary error source being the errors in the Kalman filter estimates of the clocks $e_{\text{xc}}(t_m)$. Similarly, the raw phase in Equation (32) yields a direct measure of the TCXO response, with the largest error source being the phase thermal noise $v_x(t_m)$. These results represent baseline performance curves for comparing the range measurements and can be used to ascertain the effect of specific errors introduced in the range processing, notably $e_x(t_m)$ and $e_c(t_m)$ ($e_x(t_m)$ has a much lower magnitude than either $v_x(t_m)$ or $e_c(t_m)$ and thus is not observable). Because these errors can be classified as white phase noise, the 1-s OAD values provide the best measure of the magnitude of their standard variances. Indeed, it can be shown that, for white phase noise, the following relationship holds between the standard variance $\sigma_v^2$ and the 1-s Allan variance $\sigma_y^2(1)$:

$$\sigma_v^2 = \frac{\sigma_y^2(1)}{3},$$

white phase noise (77)

Turning to the Iris TCXO range 1-s OAD, the range thermal noise $v_x(t_m)$ from the STL will be the dominant error source contributing to the OAD value of $2.2 \times 10^{-9}$. Compared with the TCXO OAD of $3.9 \times 10^{-11}$, which is smaller by a factor of 56, it can be concluded that the thermal range noise is the primary contributor to the Iris TCXO range 1-s OAD. Based on Equation (77), the thermal noise error converted to distance is 0.38 m, which is consistent with the range uncertainty calculation for $\sigma_{R_{\text{STL}}}$ from Equation (76), which also yields 0.38 m. Turning to the CSAC range, $v_x(t_m)$ is present as well as the quantization error $e_c(t_m)$ induced by the combined effect of the TCXO sample rate and the 1PPS. Recall that this term has an uncertainty of $c = 1.73$ m. Combining these two errors in an RSS yields a total range uncertainty of 1.77 m, which, when compared with the CSAC 1-s OAD of $1.28 \times 10^{-8}$ and converted to range, has a value of 2.21 m. The two values are commensurate; however, the difference is notable and will be investigated further with additional testing. Nevertheless, the theoretical range error result is useful for assessing noise levels for various signal configurations. For instance, in flight, it is expected that CAPSTONE will use a higher STL bandwidth and likely have a lower range power. In a scenario with $B_{\text{STL}} = 200$ Hz and $P_r/N_0 = 50.8$ dB-Hz, the resultant STL uncertainty is $\sigma_{R_{\text{STL}}} = 11.3$ m, clearly dominating over the quantization effect. In contrast to the one-way range rate results obtained via Method 1, which are dominated by quantization effects, and the results obtained via Method 2, which are dominated by Kalman filter effects, the overall one-way range accuracy is impacted more significantly by the effective signal power relative to the range rate.

On longer time scales, e.g., after 300 s for the Iris TCXO range or after 5,000 s for the CSAC range, systematic drift and random walk effects begin to dominate the range data. However, their effect on the CSAC range is much less significant relative to the Iris TCXO range, as shown by the detrended range results in Figure 19. Over the course of the day-long test data set, the CSAC range exhibited a total variation of $\pm 6$ m whereas the Iris TCXO range varied from $-500$ m to $+1500$ m. The superior stability of the CSAC will have a direct effect on improving orbit determination results despite exhibiting higher 1-s noise relative to the Iris TCXO range (when high signal power is used). For lower-signal-power conditions, the 1-s noise performance of the two range types will be commensurate, as thermal noise effects will dominate over quantization effects.
CONCLUSIONS

Two methods have been developed for computing calibrated one-way range rate data from an Iris radio configured with a CSAC 1PPS input. The first method is a direct calibration, but the results show that quantization effects dominate the range rate statistics. The second method begins by estimating a combined TCXO/CSAC clock phase and uses these estimates to calibrate the range rate. The second method produces more accurate results than the first method on all time scales. Indeed, it is shown that the second method yields range rate statistics approaching those that would be expected if the Iris radio were using the CSAC as its native local oscillator. For a 60-s integration time, which is typical of the integration times used for orbit determination, the first method yields a range rate uncertainty of 41 mm/s whereas the second method gives an uncertainty of 11 mm/s. Furthermore, the second method produces range rate statistics that begin to match Gaussian white noise on shorter integration times compared with the first method. Therefore, it is recommended that the second method be utilized for one-way range rates measured by an Iris/CSAC configuration in order to obtain the most accurate results with statistics that better conform to those needed for orbit determination.

The one-way range data precision spans from 0.38 m to 2.21 m (1) and is dominated by thermal noise for the Iris TCXO range and quantization error for the CSAC range. The range data quality is improved by using the CSAC 1PPS on time scales past 200 s versus the radio’s TCXO-based local Iris time. The CSAC-based range does not exhibit any significant long-term drifts over the data collection span; in contrast, it is evident that the TCXO-based range begins to wander as a result of the TCXO drift and random walk. However, maintaining this level of range precision requires a significant received range power; indeed, maintaining lock required a power >50 dB-Hz and achieving the measured precision required a power >60 dB-Hz. For one-way range at interplanetary distances, this level of received power would be difficult to achieve. One future option for improving the radio’s range processing is to consider the use of a digital phase-locked loop for symbol timing (vs. the current linear data-transition tracking loop), as it would significantly reduce the range error at lower power levels. The orbit determination processing will require simultaneous orbit and clock estimation (as expected), and the current analysis indicates that the clock estimation with a range data precision of 0.38–2.21 m is sufficient to recover either the Iris or CSAC clock behaviors (depending on which range data are being processed). This result suggests that this precision will also be sufficient for recovering a spacecraft orbit, and CSAC’s superior long-term stability should aid in minimizing orbit determination solution errors. As of September 2023, the CAPSTONE mission is in its target NRHO and, as part of its demonstration mission, is collecting one-way uplink Doppler and range data using the Iris/CSAC configuration reported in this paper. A future paper will examine these flight data and report on those findings.

ACKNOWLEDGEMENTS

The authors would like to thank John Baker for his continuing support in advancing the state of the art in small satellite navigation and for supporting this work. We also thank Matthew Chase, Scott Bryant, and Jon Streit for testing Iris’s one-way radiometric tracking capabilities. Finally, we thank Advanced Space, particularly Michael Thompson and Alec Forsman, for their vision in manifesting a test of one-way radiometric-based navigation using Iris and a CSAC on NASA’s CAPSTONE mission to the Moon. The work described in this paper was carried out in part at JPL, California Institute of Technology, under a contract with NASA (80NM0018D0004).
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How to cite this article: Ely, T., & Towfic, Z. (2024). Formulation and characterization of one-way radiometric tracking with the Iris radio using a chip-scale atomic clock. NAVIGATION, 71(1). https://doi.org/10.33012/navi.633

APPENDIX A | IRIS CARRIER PHASE NOISE VARIANCE

The Iris CTL is a multi-rate phase-locked loop (PLL) employing a loop filter that can be commanded to be either second- or third-order and that performs phase tracking and forms an estimate of the total count phase $\hat{\phi}_T(t_d(j))$. A block diagram of the CTL is shown in Figure 21. In this appendix, we develop the CTL transfer function with sufficient accuracy to form expressions for the phase noise variance $\sigma^2_{\phi_T}$, which is the sum of the variance for the received AWGN process $\sigma^2_{\phi_T',x'_d}$ and the variance due to phase noise internal to the CTL contributed by the oscillator $\sigma^2_{\phi_T',x'}$. We also develop the form of the signal that is sent to the STL for range demodulation.

Fundamental to the loop operation is the error signal $e(j)$ that is formed with the phase detector by multiplying $d(j)$ with the complex sine/cosine output of the NCO as follows:

$$e(j) = d(j)e^{-j\hat{\phi}_T(t_d(j))}$$

(78)
The Q-arm (imaginary) branch of the CTL is used to drive the loop (and route the signal to the ranging demodulator of the STL). Expanding the imaginary component of Equation (78) produces the following:

\[
\text{Im}\{e(j)\} = \frac{\sqrt{2P_\tau}}{2} \sin\left[\Delta \phi \left(t_q(j)\right) + \beta \Phi_R \left(t_q(j) - \tau \left(t_q(j)\right)\right)\right] + v_e(j) \tag{79}
\]

where, by noting \(\hat{\Phi}_R(t_q(j)) - \Phi_{R,0} = -\Delta \phi(t_q(j)) + 2\pi N_\phi\) and performing some algebraic manipulation, we can express the additive noise \(n(j)\) output by the phase detector as follows:

\[
v_e(j) = \frac{\sqrt{2}}{2} v_q(2j+1) \cos\left(\phi_{g,0} + \Delta \phi(t_q(j)) - 2\pi N_\phi\right) - \frac{\sqrt{2}}{2} v_j(2j+1) \sin\left(\phi_{g,0} + \Delta \phi(t_q(j)) - 2\pi N_\phi\right) \tag{80}
\]

For a locked loop, we have \(\Delta \phi(t_q(j)) \approx 0\) and \(d\Delta \phi / dt \approx 0\); therefore, the argument of the sine and cosine in Equation (80) can be considered nearly constant and independent of \(v_q(2j+1)\) and \(v_j(2j+1)\) (Meyr & Ascheid, 1990). With this assumption, the discrete autocorrelation of \(v_e(j)\) can be computed as follows:

\[
R_{v_e}(l) = \frac{1}{2} \left[ R_{v_q}(l) \cos^2(\cdot) + R_{v_j}(l) \sin^2(\cdot) \right] = \frac{1}{2} \left( \frac{N_0}{2T_s} \right) \delta(l) \left( \cos^2(\cdot) + \sin^2(\cdot) \right) \tag{81}
\]

and the associated two-sided discrete power spectral density is \(S_{v_e}(e^{j2\pi fT_s}) = N_0/(4T_s)\) or, in its equivalent continuous form, \(S_{v_e}(f) = N_0/4\). The signals transmitted to Iris use NRZ coding; therefore, \(\Phi_R(t) = \pm 1\) for all \(t\), and we have \(\cos(\beta \Phi_R(\cdot)) = \cos(\beta)\) and \(\sin(\beta \Phi_R(\cdot)) = \sin(\beta) \Phi_R(\cdot)\). Additionally, with \(\Delta \phi \approx 0\), Equation (79) can be linearized to yield the following:

\[
\text{Im}\{e(j)\} \approx \frac{\sqrt{2P_\tau}}{2} \Delta \phi \left(t_q(j)\right) + \frac{\sqrt{2P_\tau}}{2} \Phi_R \left(t_q(j) - \tau \left(t_q(j)\right)\right) + v_e(j) \tag{82}
\]

where, using the modulation index \(\beta\), we define the following:

\[
P_c \triangleq P_\tau \cos^2 \beta \\
P_r \triangleq P_\tau \sin^2 \beta \tag{83}
\]

as the carrier power and ranging power, respectively. The signal in Equation (79) (and its approximate linear equivalent in Equation (82)) is routed to the ranging demodulator and is discussed further in Appendix B. For analyzing the CTL, another form of \(\text{Im}\{e(j)\}\) proves useful; in particular, by rearranging Equation (82), we obtain the following:

\[
\text{Im}\{e(j)\} \approx \frac{\sqrt{2P_c}}{2} \left[ \Delta \phi \left(t_q(j)\right) + \tan \beta \Phi_R \left(t_q(j) - \tau \left(t_q(j)\right)\right) + v_e(j) \right] \tag{84}
\]
Following the same argument that led to the result in Equation (81), Equation (84) yields a noise $v_q'(j)$ that has a rescaled discrete power spectral density of $S_{v_q'}(e^{j2\pi fT_s}) = N_0/(2\,P_c^j)$ or, in its equivalent continuous form, $S_{v_q'}(f) = N_0/(2\,P_c)$. Finally, for the purpose of analyzing the CTL, the ranging/data signal $\Phi_R(t_q)$ can be considered a noise source. For convenience, we lump these additive noise sources into $v_q'(j)$ for later analysis:

$$v_q'(j) \triangleq v_q(j) + \tan \beta \Phi_R \left( t_q(j) - \tau \left(t_q(j)\right) \right)$$  \hspace{1cm} (85)

Between the phase detector and the AGC is an I&D operation that forms an average of the error $e(j)$ at a frequency of $f_K = 8$ kHz and, in the time domain, constructs the signal $\langle e(k) \rangle$ as follows:

$$\langle e(k) \rangle = \frac{1}{K} \sum_{j=(k-1)K+1}^{kK} e(j)$$  \hspace{1cm} (86)

where $K \triangleq f_s / 2f_K = 3125$. In the discrete z-domain, the associated transfer function takes the following form (Gardner, 2005):

$$F_{ID}(z^K) = \frac{1 - z^{-K}}{K(1 - z^{-1})}$$  \hspace{1cm} (87)

Note that the z transform variable is associated with discrete signals that have $T_d$ sampling and the variable $z^K$ is for the signals in the slower portion of the CTL (primarily the loop filter) that have been down-sampled to $T_K = KT_d$. The purpose of the I&D step is to reduce the computational rate requirements of the CTL, remove any subcarrier that might be present, and reduce the noise upon which the loop filter is operating (i.e., $v_q'(j)$) by a factor of $K$. Next, the AGC unitizes the coefficient in front of Equation (84) as follows:

$$\text{Im}\left\{\langle e(k) \rangle\right\} = \frac{2}{\sqrt{2P_c}} \text{Im}\{|e(k)\}|$$  \hspace{1cm} (88)

This yields a signal that is essentially the average phase error $\langle \Delta \phi(t_q(j)) \rangle$ plus some noise. The signal $\text{Im}\{|e(k)\}|$ is in a form that can be sent to the loop filter. Iris has both type 2 and 3 filters that can be selected and commanded based on the operational scenario. The type 3 filter and its resulting third-order closed-loop function are amenable to situations with high dynamics; however, the type 2 filter (and second-order closed loop) is more traditional for analysis and design. Indeed, as shown by Meyr & Ascheid (1990), filters with a dominant pole pair are well approximated by a second-order closed-loop system transfer function. This is the case for Iris; thus, we focus on analyzing a type 2 filter consisting of proportional and integral feedback, with the time output given as follows:

$$u(k) = K_1 \text{Im}\{|\bar{e}(k)\}| + K_2 \sum_{p=0}^{k} \text{Im}|\bar{e}(p)|$$  \hspace{1cm} (89)

and an associated transfer function that takes the following form:

$$F_L(z^K) = K_1 + \frac{K_2}{1 - z^{-K}}$$  \hspace{1cm} (90)
The output of the loop filter is now sent to the NCO; however, because the NCO is still operating at 25 MHz, there is effectively a zero-order hold on the input signal that up-samples it from \( T_k \) to \( T_d \). Consequently, the NCO integrates the same loop filter output \( K \) times before advancing to the next loop filter output at time \( k + 1 \).

In the time domain, the hold signal is represented as follows:

\[
v(j) = \frac{1}{K} u(k) \quad \text{for} \quad kK < j \leq (k+1)K
\]

where the division by \( K \) normalizes the output so that the NCO yields the correct phase after integrating \( K \) times. As shown by Gardner (2005), the transfer function for the hold with normalization by \( 1/K \) is given by the following:

\[
F_H(z) = \frac{1}{K} \frac{1 - z^{-K}}{1 - z^{-1}}
\]

(92)

The NCO integrates \( v(j) \) and advances the phase according to the following:

\[
\hat{\phi}_T(t_j(j+1)) = \hat{\phi}_T(t_j(j)) + v(j)
\]

(93)

and has the following associated transfer function:

\[
F_{NCO}(z) = \frac{z^{-1}}{1 - z^{-1}}
\]

(94)

Note that the NCO is driven by the local oscillator; thus, additional noise from the oscillator \( \psi_I(t_j) \) is introduced at this point (see Figure 21) in the tracking loop (Meyr & Ascheid, 1990). This oscillator noise appears in conjunction with the additive noise in Equation (85) and the oscillator error introduced previously when the carrier signal was mixed down to baseband (i.e., the presence of \( x_I(t) \) in Equation (10)). The final step in the CTL prior to closing the loop is the sine and cosine lookups that are used to generate the signal \( e^{-j\hat{\phi}_T(t_j(j+1))} \). The CTL closes the loop at the phase detector with the DDC signal \( d(j+1) \), which has now advanced to the next time step \( T_d = 2T_s \) seconds later.

The noise sources in the CTL, \( v'_I(t_j) \) and \( \psi_I(t_j) \), are acted upon and “shaped” by the CTL via closure of the loop. To facilitate this analysis, we form the open-loop transfer function and then close it. The open-loop transfer function is formed via Equations (94), (92), (90), and (87) as follows:

\[
G_d(z) = \frac{\dot{\phi}_T(z)}{\epsilon(z)} = F_{NCO}(z) F_H(z) F_I(z^K) F_{ID}(z^K)
\]

\[
= \left( \frac{z^{-1}}{1 - z^{-1}} \right) \left( 1 - z^{-K} \right) \left( \frac{K_1 + K_2}{K} \right) \left( \frac{1 - z^{-K}}{K} \right)
\]

(95)

where the effect of normalization by the AGC has been implicitly included. The closed-loop transfer function can now be obtained in the standard way:

\[
H_d(z) = \frac{\dot{\phi}_T(z)}{\phi_T(z)} = \frac{G_d(z)}{1 + G_d(z)}
\]

(96)

When the CTL is tracking, its operating noise-equivalent loop bandwidth \( B_{CTL} \) is narrow compared with the input rate \( f_d = 1/T_d \) or even the loop rate
\( f_K = 1/T_K = 1/KT_d \); as a result, it is practical to approximate the discrete transfer function \( G_d(z) \) with its approximate continuous counterpart \( G(s) \). Indeed, noting that \( z = e^{sT_d} \approx 1 + sT_d + \ldots \) and assuming \( B_nT_d < B_nT_K \ll 1 \), the following first-order approximations prove useful:

\[
\begin{align*}
\frac{1}{z} & \approx \frac{1}{1 - z^{-1}} \approx z^{-K} \approx 1 \\
1 - \frac{1}{z} & \approx sT_d \\
1 - z^{-K} & \approx sT_K = sKT_d
\end{align*}
\] (97)

By substituting the approximations from Equation (97) into Equation (95) and simplifying, we obtain the following:

\[
G_d(s) \approx \left( \frac{1}{sT_d} \right) \left( \frac{1}{K} \frac{sKT_d}{sT_d} \right) \left( K_1 + \frac{K_2}{sT_K} \right) = \left( \frac{K_1}{sT_K} \right) \left( \frac{K_1 + K_2}{sT_K} \right), \quad B_{CTL}T_K \ll 1
\] (98)

which matches the typical result for a continuous PLL with proportional and integral control. Next, we substitute Equation (98) into Equation (96) and reduce to obtain the following:

\[
H_d(s) \approx \frac{KK_1}{T_K S + \frac{KK_1}{T_K^2} + \frac{KK_2}{T_K^2}}, \quad B_{CTL}T_K \ll 1
\] (99)

Comparing Equation (99) with the standard second-order PLL result gives the following relation:

\[
H(s) = \frac{2\zeta \omega_n S + \omega_n^2}{S^2 + 2\zeta \omega_n S + \omega_n^2}
\] (100)

where \( \zeta \) is a non-dimensional damping coefficient and \( \omega_n \) is the natural frequency in rad/s. We can then make the following associations:

\[
K_1 = \frac{2\zeta}{K} \left( \omega_n T_K \right) = \frac{1}{K} \left( \frac{4\zeta}{\zeta + \frac{1}{4\zeta}} \right) (B_{CTL}T_K), \quad B_{CTL}T_K \ll 1
\] (101)

\[
K_2 = \frac{1}{K} \left( \omega_n T_K \right)^2 = \frac{1}{K} \left[ \left( \zeta + \frac{1}{4\zeta} \right)^2 \right] (B_{CTL}T_K)^2, \quad B_{CTL}T_K \ll 1
\]

where we used the following relation:

\[
B_{CTL} = \frac{\omega_n}{2} \left( \zeta + \frac{1}{4\zeta} \right) \text{(Hz)}
\] (102)

Therefore, we can use the continuous closed-loop transfer function \( H(s) \) as a proxy for the discrete CTL function \( H_d(z) \) to analyze its noise properties when the loop is locked and \( B_{CTL}T_K \ll 1 \). To facilitate this, we rewrite Equation (100) by substituting \( s = \frac{j2\pi f}{f_n} \) and reducing:

\[
H(f) = \frac{f_n^2 + \frac{3}{2} \zeta f_n f}{f_n^2 - f^2 + \frac{3}{2} \zeta f_n f}
\] (103)
where $\omega_n = 2\pi f_0$. Using this expression, the phase error variance $\sigma^2_{\phi}$ is induced by

$$
\nu'_g(t) \text{ (the continuous counterpart to the discrete noise defined in Equation (85)),}
$$

and $\psi_I(t)$ is determined according to the following result from Meyr & Ascheid (1990):

$$
\sigma^2_{\phi} = \int_{-\infty}^{\infty} |H(f)|^2 S_{\nu'_g}(f) df + \int_{0}^{\infty} [1 - H(f)]^2 S_{\psi_I}(f) df \tag{104}
$$

where $S_{\nu'_g}(f)$ is the two-sided spectral density of $\nu'_g(t)$ and $S_{\psi_I}(f)$ is the one-sided spectral density of $\psi_I(t)$. We now examine the three noise processes contributing to Equation (104):

1. As noted previously, the noise spectral density of $\nu'_g(t)$ is

$$
S_{\nu'_g}(f) = N_0 / (2P_c) = N_0 / (2P_c \cos^2 \beta).
$$

2. As discussed by Lesh (1978) for a chip rate that is large relative to the loop bandwidth (i.e., $B_{CTL} T_r \ll 1$), the polar NRZ process $\Phi_R(k)$ can be approximated by a white noise process with the following two-sided spectral density:

$$
S_{\phi_g}(f) = T_r \text{sinc}^2(f T_r) \approx T_r, \quad B_{CTL} T_r \ll 1 \tag{105}
$$

Hamkins et al. (2016) showed that the range codes have a correlated spectrum; thus, they do not yield a truly random and uncorrelated sequence. However, their results also showed that phase error variances induced by Equation (105) are an upper bound to those induced by the range codes. Therefore, we will use Equation (105) as a conservative bound for the effect of the range signal on the phase error variance.

3. The oscillator phase noise $\psi_I(t)$ of the Iris TCXO (Q-Tech 804) has the following specified single sideband power spectral density $L^1_{\psi_I}(f)$ profile at $f_0$, which can be associated with the one-sided phase spectral density function $S^1_{\psi_I}(f)$ (again, using the form given by Riley & Howe (2008)) as follows:

$$
\begin{align*}
L^1_{\psi_I}(f) &= \begin{cases} 
-80 \text{dBc/Hz}, f = 10 \text{ Hz} \\
-110 \text{dBc/Hz}, f = 100 \text{ Hz} \\
-135 \text{dBc/Hz}, f = 1 \text{ kHz} \\
-155 \text{dBc/Hz}, f = 10 \text{ kHz} \\
-155 \text{dBc/Hz}, f = 100 \text{ kHz}
\end{cases} \\
S^1_{\psi_I}(f) &= 2 \times 10^{-10} \left( \frac{f_0}{f} \right)^2 \left( h_2 f^2 + h_1 f + h_0 + \frac{h_{-1}}{f^2} + \frac{h_{-2}}{f^3} \right) \\
&= \left( \frac{f_0}{f} \right)^2 \left( 1.534 \times 10^{-31} + \frac{1.924 \times 10^{-23}}{f^2} + \frac{5.884 \times 10^{-21}}{f^3} + \frac{1.920 \times 10^{-19}}{f^4} \right) \tag{106}
\end{align*}
$$

Phase noise models for oscillators are often expressed as one-sided spectral densities (Riley & Howe, 2008) and have been identified with a superscript 1. This standard has been used in this paper and is the reason the second integral in Equation (104) is evaluated over only positive frequencies.
For reference, the $h_2$ term is white phase noise, the $h_1$ term is flicker phase noise, the $h_0$ term is white frequency noise, the $h_{-1}$ term is flicker frequency noise, and the $h_{-2}$ term is random walk frequency noise. The numerical values in the last line for $h_i$ were obtained by solving for the five equations and five unknowns given by the $\mathcal{L}_{\psi_i}(f)$ specification. Note that the $h_1$ term (i.e., flicker phase) is not present for the Iris TCXO as the solution produced $h_1 < 0$, which is not possible because all of the coefficients are strictly positive by definition.

Using the preceding notes, the phase noise variance for the AWGN process $n_\psi(k)$ and the data process $\Phi_R(k)$ for the first part of Equation (104) can be computed with the following result:

$$
\sigma_{\psi,\nu'}^2 = 2 \int_0^\infty |H(f)|^2 S_{\psi'}(f) df \\
= 2 \left( \frac{N_0}{2P_c \cos^2 \beta} + T_R \tan^2 \beta \right) B_{CTL} \\
= \frac{N_0 B_{CTL}}{P_c} (1 + 2R_s) \text{ (rad}^2) \tag{107}
$$

where we have defined the ranging signal-to-noise ratio as follows:

$$
R_s \equiv \frac{P_T R}{N_0} = \frac{P_T}{N_0 f_R} \tag{108}
$$

Turning to the oscillator phase noise portion of Equation (104), the Iris CTL is typically run with $B_{CTL} = 200$ Hz and $\zeta = \sqrt{2}/2 \rightarrow f_n = \frac{\sqrt{2}}{3\pi} B_{CTL} \approx 60$ Hz. Using Equation (103) and Equation (106), we arrive at the following:

$$
\sigma_{\phi,\nu'}^2 = \int_0^\infty \left[ 1 - H(f) \right]^2 S_{\phi_i}(f) df \\
= \int_0^\infty \frac{f^4}{f_n^2} \left( \frac{f_0}{f} \right)^2 \left( h_2 f^2 + h_0 + \frac{h_{-1}}{f^1} + \frac{h_{-2}}{f^2} \right) df \\
\approx f_0^2 \left( \frac{\sqrt{2} h_2}{4 f_n} + \frac{\sqrt{2} h_0}{4 f_n} + \frac{\sqrt{2} h_{-1}}{4 f_n^3} \right) \text{ (rad}^2) \tag{109}
$$

where the approximation results from integrating the white phase noise $h_2$ term. This term is limited by a high-frequency cutoff $f_h$ where we assume $f_h \gg f_n$ and can be set to the oscillator frequency $f_0$. As a representative numerical example, $B_{CTL} = 200$ Hz yields $\sigma_{\phi,\nu'} = 0.00015$ rad. The preceding variances are additive and result in an overall noise variance for the Iris carrier phase $\sigma_{\phi}^2$, which takes the following form:

$$
\sigma_{\phi}^2 = \sigma_{\phi,\nu'}^2 + \sigma_{\phi,\nu'}^2 \\
= \frac{N_0 B_{CTL}}{P_c} (1 + 2R_s) + f_0^2 \left( \frac{\sqrt{2} h_2}{4 f_n} + \frac{\sqrt{2} h_0}{4 f_n} + \frac{\sqrt{2} h_{-1}}{4 f_n^3} + \frac{\sqrt{2} h_{-2}}{4 f_n^3} \right) \text{ (rad}^2) \tag{110}
$$
APPENDIX B  | IRIS SYMBOL TIMING VARIANCE

The analysis in this appendix develops an expression for \( \sigma^2_{\varepsilon_r} \), the variance uncertainty in the symbol timing estimate \( \hat{\varepsilon}_r \) generated by the Iris STL. The STL is based on the linear data-transition tracking loop (LDTTL) developed by Simon (2005); however, the Iris timing error detector is different from that of Simon and results in a slightly different timing error variance. A block diagram for the Iris STL is presented in Figure 22 (top), and its equivalent continuous linear representation in the s-domain is shown in Figure 22 (bottom). The analysis for the timing error variance induced by the STL is presented in an abbreviated form that follows the outline of the original DTTL analysis (Simon, 1969) and LDTTL analysis (Simon, 2005).

We begin by converting Equation (84) into a continuous form by noting that \( T_r \gg T_s \), assuming that the carrier phase error is negligible ((\( \Delta \phi(t) \approx 0 \)), substituting Equation (62) for \( \Phi_R(t) \), and, for now, ignoring the light time delay \( \tau(t) \). This approach yields the resulting form for the input signal to the STL:

\[
\text{Im}[\varepsilon(t)] = \sqrt{\frac{2P_r}{2}} \sum_{q=-\infty}^{\infty} \Psi(q) p(t - qT_r - \varepsilon_r) + \nu_v(t)
\]

(111)

STL time-domain nonlinear model

![Diagram of Iris STL used for ranging demodulation in the discrete time domain (top) and a continuous linear representation in the s-domain where VCO is a voltage controlled oscillator (bottom)](imagetypings)

STL equivalent linear model

![Diagram of Iris STL equivalent linear model](imagetypings)
Recall that the double-sided power spectral density is \( S_n(f) = N_0/4 \), with an autocorrelation of \( R_n(h) = (N_0/4)\delta(h) \) and variance \( \sigma_n^2 = N_0/4 \).

The structure of the STL is designed to send the ranging signal in Equation (111) into two branches that include I&D filters that depend on the estimate \( \hat{\epsilon}_r \) of \( \epsilon_r \). For reference, we define \( \hat{\epsilon}_r \approx (e_r + \epsilon_r) \), where the noise process \( \epsilon_r \) has variance \( \sigma_r^2 / 2 \) and represents the symbol timing error of the STL. Similar to the CTL, the I&D filters primary function are to reduce the sampling rate via accumulation over a specified interval. For the STL, the reduced sampling rate is the chip rate. The I&D filters also assist in forming the STL error signal, which is then filtered and fed back to the I&D filters, leading to closed-loop tracking of \( \hat{\epsilon}_r \) and the generation of soft symbols \( \Psi(k) \).

The I&D filters yield signals that are out of phase with each other and can be combined to form a timing error signal \( e(k) \) that is then filtered and integrated via an NCO to obtain the timing estimate \( \hat{\epsilon}_r \).

Fundamental assumptions for the analysis include the following:

- Similar to the CTL, we assume that the STL noise equivalent bandwidth \( B_{STL} \) is much less than the the chip rate, that is, \( B_{STL}T_r \ll 1 \). With this assumption, we can employ a continuous analysis of the STL and assume that both \( \hat{\epsilon}_r \) and \( \epsilon_r \) are nearly constant over a significant number of ranging chips. Indeed, both \( \frac{d\epsilon_r}{dt} \) and \( \frac{d\hat{\epsilon}_r}{dt} \) are negligible.

- We define the normalized symbol timing error as \( \hat{\lambda} = \frac{\epsilon_r}{T_r} = \left( e_r - \hat{\epsilon}_r \right)/T_r \) with range \(-\frac{1}{2} \leq \hat{\lambda} \leq \frac{1}{2} \). We assume that the STL has acquired and is tracking \( \epsilon_r \); therefore, the approximation of \( \hat{\lambda} \approx 0 \) and a linear analysis of the STL are valid.

The outputs of the in-phase and mid-phase I&D step are given by the following:

\[
\begin{align*}
\text{in} & \quad y_{in}(k) = K_1 \int_{(k-1)T_r + \hat{\epsilon}_r}^{kT_r + \hat{\epsilon}_r} \text{Im}(e(t))dt = K_1 \int_{(k-1)T_r + \hat{\epsilon}_r}^{kT_r + \hat{\epsilon}_r} s(t, \epsilon_r)dt + K_1 \int_{(k-1)T_r + \hat{\epsilon}_r}^{kT_r + \hat{\epsilon}_r} \epsilon_r(t)dt \\
\text{mid} & \quad y_{mid}(k) = K_2 \int_{\left\lceil \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r}^{\left\lfloor \frac{k+1}{2} \right\rfloor T_r + \hat{\epsilon}_r} \text{Im}(e(t))dt = K_2 \int_{\left\lceil \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r}^{\left\lfloor \frac{k+1}{2} \right\rfloor T_r + \hat{\epsilon}_r} s(t, \epsilon_r)dt + K_2 \int_{\left\lceil \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r}^{\left\lfloor \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r} \epsilon_r(t)dt
\end{align*}
\]

where we have defined the output of the in-phase integrator to be “soft symbols” \( \Psi(k) \) that represent a noisy form of the range signal that will be correlated upon. \( k \) is the time index of output signals from the I&D operation, which advance by unity every chip duration \( T_r \). Because the intervals that define \( v(k) \) and \( \mu(k) \) overlap, these noise processes are not independent; thus, it is useful to separate them into independent processes as follows:

\[
\begin{align*}
v(k) &= \int_{(k-1)T_r + \hat{\epsilon}_r}^{kT_r + \hat{\epsilon}_r} n(t)dt + \int_{\left\lfloor \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r}^{\left\lceil \frac{k}{2} \right\rceil T_r + \hat{\epsilon}_r} n(t)dt \\
\mu(k) &= \int_{kT_r + \hat{\epsilon}_r}^{(k+1)T_r + \hat{\epsilon}_r} n(t)dt + \int_{\left\lfloor \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r}^{\left\lceil \frac{k+1}{2} \right\rceil T_r + \hat{\epsilon}_r} n(t)dt
\end{align*}
\]
where we make the following observations about $N(k)$ and $M(k)$:

1. $N(k)$, $M(n)$ are mutually independent for all $k$, $n$.
2. $N(k)$, $N(n)$ and $M(k)$, $M(n)$ are mutually independent for all $k \neq n$.

Furthermore, we find the discrete noise strength for $N(k)$ and $M(k)$:

$$
\sigma^2_N = \sigma^2_M = \frac{N_0 T_r}{8}
$$

(114)

Expressions for $c(k)$ and $b(k)$ are obtained by assuming piecewise constant values for the components of $s(t, \varepsilon_r)$ with the following result:

$$
c(k) = K_1 \sqrt{2P_r} T_R \left( \Psi(k-2) + \Psi(k-1)(1-\lambda) \right), \quad 0 \leq \lambda \leq \frac{1}{2}
$$

$$
b(k) = K_2 \sqrt{2P_r} T_R \left( \Psi(k-1) \left( \frac{1}{2} + \lambda \right) + \Psi(k) \left( \frac{1}{2} - \lambda \right) \right), \quad 0 \leq \lambda \leq \frac{1}{2}
$$

(115)

The timing error detector combines $y_{in}(k)$ and $y_{mid}(k)$ to form an error signal $e(k)$ as follows:

$$
e(k) = y_{in}(k) \left( y_{mid}(k) - y_{mid}(k-1) \right)
$$

(116)

This result differs from the LDTTL error signal of Simon, and as a result, the S-curve and associated timing error variance differ. The S-curve is the statistical average of the error signal $e(k)$ and is defined as the conditional expectation of $e(k)$:

$$
g(\lambda) \triangleq E[e(k) | \lambda]
$$

(117)

This can be evaluated in a straightforward manner if we make a simplifying assumption for $\Psi(k) = \pm 1$ that each value is equiprobable\(^5\) (i.e., $p_{\Psi}(1) = p_{\Psi}(-1) = \frac{1}{2}$) and each instance of $\Psi(k)$ is independent from the other. By substituting Equation (115) into Equation (116) and performing the expectation in Equation (117), we obtain the following result:

$$
g(\lambda) = \frac{3}{4} K_1 K_2 P_r T_r^2 \left( -\frac{1}{2} \lambda - \text{sgn}(\lambda) \lambda^2 \right), \quad -\frac{1}{2} \leq \lambda \leq \frac{1}{2}
$$

(118)

where the interval was extended to the full range $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$ based on the fact that $g(\lambda)$ is an odd function (i.e., $g(\lambda) = -g(-\lambda)$). Similar to the analysis for the CTL, we assume that the STL is tracking and that $\lambda \approx 0$; therefore, $g(\lambda)$ can be approximated as follows:

$$
g(\lambda) \approx \left[ \frac{dg}{d\lambda} \right]_{\lambda=0} \lambda
$$

$$
= \frac{3}{8} \left( K_1 K_2 P_r T_r^2 \right) \lambda
$$

(119)

\(^5\) Recall that the JPL or T4B PN sequence values are not entirely equiprobable (Massey et al., 2007; Consultative Committee for Space Data Systems, 2022); however, because of the dominance of the range clock in the sequence, this simplifying approximation is useful in it that provides a conservative estimate of the variance of the timing error.
The error signal \( e(k) \) can be characterized as the sum of \( g(\lambda) \) and a noise process \( n_\lambda(t) \), which, with some rearranging, yields the following definition:

\[
n_\lambda(t) \triangleq e(k) - g(\lambda)
\]

(120)

The process \( n_\lambda(t) \) represents an additive noise that defines the variation of \( e(k) \) about its mean \( g(\lambda) \) and is piecewise constant (because of the I&D operation) over sample intervals. To characterize the statistics of \( n_\lambda(t) \), we find the autocorrelation function \( R_{n_\lambda}(h) \) that is evaluated on sample values \( h = mT_r \) via the following:

\[
R_{n_\lambda}(h) = E[n_\lambda(t) n_\lambda(t + mT_r) | \lambda]
= E[(e(k) - g(\lambda))(e(k + m) - g(\lambda)) | \lambda]
= E[e(k)e(k + m) | \lambda] - g^2(\lambda) \triangleq R_{n_\lambda}(m, \lambda), \quad m = 0, \pm 1, \pm 2, \ldots
\]

(121)

where, to simplify notation, we have dropped the conditional expectation on \( \lambda \).

The two-sided power spectral density can now be formed:

\[
S_{n_\lambda}(f) = \int R_{n_\lambda}(h)e^{-i2\pi fh}dh
= T_r \sum_{m=-\infty}^{\infty} R_{n_\lambda}(m, \lambda)e^{-i2\pi fh}
\approx T_r \sum_{m=-\infty}^{\infty} R_{n_\lambda}(m, \lambda), \quad B_{STL}T_r \ll 1
\]

(122)

In the last step of Equation (122), we note that the spectral density is approximately flat over frequency (hence, it has a constant value), and following the usual convention, we define the double-sided spectral density using the following:

\[
\frac{N_0'}{2} \triangleq S_{n_\lambda}(f) \rightarrow N_0' \approx 2T_r \left( R_{n_\lambda}(0, \lambda) + 2 \sum_{m=1}^{\infty} R_{n_\lambda}(m, \lambda) \right), \quad B_{STL}T_r \ll 1
\]

(123)

where we have used the fact that the component autocorrelation functions are even, i.e., \( R_{n_\lambda}(m, \lambda) = R_{n_\lambda}(-m, \lambda) \). Similar to the analysis for the CTL, we assume that the STL is tracking and that \( \lambda \approx 0 \). By expressing Equation (123) as expectations and applying \( \lambda \approx 0 \), we obtain the following:

\[
N_0' \approx 2T_r \left( E[e^2(k) | \lambda = 0] + 2 \sum_{m=1}^{\infty} E[e(k)e(k + m) | \lambda = 0] \right), \quad B_{STL}T_r \ll 1
\]

(124)

After a lengthy amount of algebra, the following can be demonstrated:

\[
R_{n_\lambda}(0, 0) = E[e^2(k) | \lambda = 0] = \frac{1}{8} (K_1K_2P_2T_r^2)^2 \left( 1 + \frac{5}{2R_s} + \frac{1}{R_s^2} \right)
\]

\[
R_{n_\lambda}(1, 0) = E[e(k)e(k + 1) | \lambda = 0] = -\frac{1}{64} (K_1K_2P_2T_r^2)^2 \left( \frac{1}{2} + \frac{1}{R_s} \right)^2
\]

(125)

\[
R_{n_\lambda}(m, 0) = E[e(k)e(k + m) | \lambda = 0] = 0, \quad m \neq 0, 1
\]
By substituting the expressions in Equation (125) into Equation (124) and reducing, we obtain the following:

$$N'_0 = T_r \left(K_r K_2 P_r T_e^2\right) \left(\frac{1}{2R_s} \left(1 + \frac{3}{8R_s}\right)\right)$$  \(126\)

where we have defined $R_s \equiv P_r T_e / N_0$.

Because $\lambda \approx 0$, we can approximate the STL response using linear methods. The equivalent linear representation of the STL shown in Figure 4 (bottom) and the transfer function relationship in the s-domain between $\lambda(s)$ and $n_s(s)$ can be obtained and, with the assumption of $de_r/dt \approx 0 \rightarrow se_r(s) \approx 0$, takes the following form:

$$\lambda(s) = \frac{K_g F_r(s)}{s + K_g F_L(s)} n_s(s) \frac{n_s(s)}{K_g}$$  \(127\)

$$= H_s(s) \frac{n_s(s)}{K_g}$$

where the closed-loop transfer function of the STL is $H_s(s)$. We define the STL noise equivalent loop bandwidth in the same fashion as we did with the CTL to yield the $B_{STL}$ in hertz. Furthermore, the power spectral density of $n_s(s) / K_g$ is $N'_0 / (2K_g^2)$. In this fashion, we can use the result of Equation (104) to show that the variance $\sigma^2_\lambda$ and, equivalently, the timing error variance $\sigma^2_{\varepsilon_r}$ can be found as follows:

$$\sigma^2_\lambda = \frac{N'_0 B_{STL}}{K_g^2} \rightarrow \sigma^2_{\varepsilon_r} = \frac{T_r^2 N'_0 B_{STL}}{K_g^2}$$  \(128\)

Substituting Equation (126) and Equation (119) into Equation (128) yields the following expression for the symbol timing error variance $\sigma^2_{\varepsilon_r}$ of the STL:

$$\sigma^2_{\varepsilon_r} = T_r^2 \left[\frac{N'_0 B_{STL}}{2P_r} \left(1 + \frac{3}{8R_s}\right)\right]$$  \(129\)